

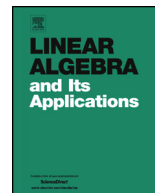


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## The star-shapedness of a generalized numerical range



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### ABSTRACT

Let  $\mathcal{H}_n$  be the set of all  $n \times n$  Hermitian matrices and  $\mathcal{H}_n^m$  be the set of all  $m$ -tuples of  $n \times n$  Hermitian matrices. For  $A = (A_1, \dots, A_m) \in \mathcal{H}_n^m$  and for any linear map  $L : \mathcal{H}_n^m \rightarrow \mathbb{R}^\ell$ , we define the  $L$ -numerical range of  $A$  by

$$W_L(A) := \{L(U^*A_1U, \dots, U^*A_mU) : U \in \mathbb{C}^{n \times n}, U^*U = I_n\}.$$

In this paper, we prove that if  $\ell \leq 3$ ,  $n \geq \ell$  and  $A_1, \dots, A_m$  are simultaneously unitarily diagonalizable, then  $W_L(A)$  is star-shaped with star center at  $L\left(\frac{\text{tr}A_1}{n}I_n, \dots, \frac{\text{tr}A_m}{n}I_n\right)$ .

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## 1. Introduction

Let  $\mathbb{C}^{n \times n}$  denote the set of all  $n \times n$  complex matrices, and  $A \in \mathbb{C}^{n \times n}$ . The (classical) numerical range of  $A$  is defined by

$$W(A) := \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

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The properties of  $W(A)$  were studied extensively in the last few decades and many nice results were obtained; see [10,13]. The most beautiful result is probably the Toeplitz–Hausdorff Theorem which affirmed the convexity of  $W(A)$ ; see [12,17]. The generalizations of  $W(A)$  remain an active research area in the field.

For any  $A \in \mathbb{C}^{n \times n}$ , write  $A = A_1 + iA_2$  where  $A_1, A_2$  are Hermitian matrices. Then by regarding  $\mathbb{C}$  as  $\mathbb{R}^2$ , one can rewrite  $W(A)$  as

$$W(A) := \{(x^* A_1 x, x^* A_2 x) : x \in \mathbb{C}^n, x^* x = 1\}.$$

This expression motivates naturally the generalization of the numerical range to the joint numerical range, which is defined as follows. Let  $\mathcal{H}_n$  be the set of all  $n \times n$  Hermitian matrices and  $\mathcal{H}_n^m$  be the set of all  $m$ -tuples of  $n \times n$  Hermitian matrices. The joint numerical range of  $A = (A_1, \dots, A_m) \in \mathcal{H}_n^m$  is defined as

$$W(A) = W(A_1, \dots, A_m) := \{(x^* A_1 x, \dots, x^* A_m x) : x \in \mathbb{C}^n, x^* x = 1\}.$$

It has been shown that for  $m \leq 3$  and  $n \geq m$ , the joint numerical range is always convex [1]. This result generalizes the Toeplitz–Hausdorff Theorem. However, the convexity of the joint numerical range fails to hold in general for  $m > 3$ , see [1,11,14].

When a new generalization of numerical range is introduced, people are always interested in its convexity. Unfortunately, this nice property fails to hold in some generalizations. However, another property, namely star-shapedness, holds in some generalizations; see [5,18]. Therefore, the star-shapedness is the next consideration when the generalized numerical ranges fail to be convex. A set  $M$  is called star-shaped with respect to a star-center  $x_0 \in M$  if for any  $0 \leq \alpha \leq 1$  and  $x \in M$ , we have  $\alpha x + (1 - \alpha)x_0 \in M$ . In [15], Li and Poon showed that for a given  $m$ , the joint numerical range  $W(A_1, \dots, A_m)$  is star-shaped if  $n$  is sufficiently large.

Let  $\mathcal{U}_n$  be the set of all  $n \times n$  unitary matrices. For  $C \in \mathcal{H}_n$  and  $A = (A_1, \dots, A_m) \in \mathcal{H}_n^m$ , the joint  $C$ -numerical range of  $A$  is defined by

$$W_C(A) := \{(\text{tr}(CU^* A_1 U), \dots, \text{tr}(CU^* A_m U)) : U \in \mathcal{U}_n\},$$

where  $\text{tr}(\cdot)$  is the trace function. When  $C$  is the diagonal matrix with diagonal elements  $1, 0, \dots, 0$ , then  $W_C(A)$  reduces to  $W(A)$ . Hence the joint  $C$ -numerical range is a generalization of the joint numerical range. In [3], Au-Yeung and Tsing generalized the convexity result of the joint numerical range to the joint  $C$ -numerical range by showing that  $W_C(A)$  is always convex if  $m \leq 3$  and  $n \geq m$ . However  $W_C(A)$  fails to be convex in general if  $m > 3$ . One may consult [6] and [7] for the study of the convexity of  $W_C(A)$ . The star-shapedness of  $W_C(A)$  remains unclear for  $m > 3$ .

For  $A = (A_1, \dots, A_m) \in \mathcal{H}_n^m$ , we define the joint unitary orbit of  $A$  by

$$\mathcal{U}_n(A) := \{(U^* A_1 U, \dots, U^* A_m U) : U \in \mathcal{U}_n\}.$$

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