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The star-shapedness of a generalized numerical range

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A R T I C L E I N F O

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A B S T R A C T

Let $\mathcal{H}_n$ be the set of all $n \times n$ Hermitian matrices and $\mathcal{H}_n^m$ be the set of all m-tuples of $n \times n$ Hermitian matrices. For $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$ and for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$, we define the L-numerical range of A by

$$W_L(A) := \{L(U^*A_1U, ..., U^*A_mU) : U \in \mathbb{C}^{n\times n}, U^*U = I_n\}.$$

In this paper, we prove that if $\ell \leq 3$, $n \geq \ell$ and $A_1, ..., A_m$ are simultaneously unitarily diagonalizable, then $W_L(A)$ is star-shaped with star center at $L\left(\frac{\mathrm{tr}A_1}{n}I_n, ..., \frac{\mathrm{tr}A_m}{n}I_n\right)$.

1. Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices, and $A \in \mathbb{C}^{n\times n}$. The (classical) numerical range of A is defined by

$$W(A) := \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

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The properties of $W(A)$ were studied extensively in the last few decades and many nice results were obtained; see [10,13]. The most beautiful result is probably the Toeplitz–Hausdorff Theorem which affirmed the convexity of $W(A)$; see [12,17]. The generalizations of $W(A)$ remain an active research area in the field.

For any $A \in \mathbb{C}^{n\times n}$, write $A = A_1 + iA_2$ where A_1, A_2 are Hermitian matrices. Then by regarding $\mathbb{C}$ as $\mathbb{R}^2$, one can rewrite $W(A)$ as

$$W(A) := \{(x^*A_1x, x^*A_2x) : x \in \mathbb{C}^n, x^*x = 1\}.$$

This expression motivates naturally the generalization of the numerical range to the joint numerical range, which is defined as follows. Let $\mathcal{H}_n$ be the set of all $n \times n$ Hermitian matrices and $\mathcal{H}_n^m$ be the set of all m-tuples of $n \times n$ Hermitian matrices. The joint numerical range of $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$ is defined as

$$W(A) = W(A_1, ..., A_m) := \{(x^*A_1x, ..., x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}.$$

It has been shown that for $m \le 3$ and $n \ge m$, the joint numerical range is always convex [1]. This result generalizes the Toeplitz–Hausdorff Theorem. However, the convexity of the joint numerical range fails to hold in general for $m > 3$, see [1,11,14].

When a new generalization of numerical range is introduced, people are always interested in its convexity. Unfortunately, this nice property fails to hold in some generalizations. However, another property, namely star-shapedness, holds in some generalizations; see [5,18]. Therefore, the star-shapedness is the next consideration when the generalized numerical ranges fail to be convex. A set M is called star-shaped with respect to a star-center $x_0 \in M$ if for any $0 \le \alpha \le 1$ and $x \in M$, we have $\alpha x + (1-\alpha)x_0 \in M$. In [15], Li and Poon showed that for a given m, the joint numerical range $W(A_1, ..., A_m)$ is star-shaped if n is sufficiently large.

Let $\mathcal{U}_n$ be the set of all $n\times n$ unitary matrices. For $C \in \mathcal{H}_n$ and $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$, the joint C-numerical range of A is defined by

$$W_C(A) := \{(\mathrm{tr}(CU^*A_1U), ..., \mathrm{tr}(CU^*A_mU) : U \in \mathcal{U}_n\},$$

where $\mathrm{tr}(\cdot)$ is the trace function. When C is the diagonal matrix with diagonal elements $1, 0, ..., 0$, then $W_C(A)$ reduces to $W(A)$. Hence the joint C-numerical range is a generalization of the joint numerical range. In [3], Au-Yeung and Tsing generalized the convexity result of the joint numerical range to the joint C-numerical range by showing that $W_C(A)$ is always convex if $m \le 3$ and $n \ge m$. However $W_C(A)$ fails to be convex in general if $m > 3$. One may consult [6] and [7] for the study of the convexity of $W_C(A)$. The star-shapedness of $W_C(A)$ remains unclear for $m > 3$.

For $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$, we define the joint unitary orbit of A by

$$\mathcal{U}_n(A) := \{(U^*A_1U, ..., U^*A_mU) : U \in \mathcal{U}_n\}.$$