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Oriented gain graphs, line graphs and eigenvalues

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A B S T R A C T

A theory of orientation on gain graphs (voltage graphs) is developed to generalize the notion of orientation on graphs and signed graphs. Using this orientation scheme, the line graph of a gain graph is studied. For a particular family of gain graphs with complex units, matrix properties are established. As with graphs and signed graphs, there is a relationship between the incidence matrix of a complex unit gain graph and the adjacency matrix of the line graph.

1. Introduction

The study of acyclic orientations of a graph enjoys a rich history of remarkable results, including Greene's bijection between acyclic orientations and regions of an associated hyperplane arrangement [9], as well as Stanley's theorem on the number of acyclic orientations [21]. In [26], Zaslavsky develops a more general theory of orientation on *signed graphs* (graphs with edges labeled either $+1$ or -1), and shows that regions defined by

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a signed graphic hyperplane arrangement correspond with acyclic orientations of particular signed graphs. Zaslavsky's approach is intimately connected with the theory of oriented matroids [3].

A *biased graph* is a graph with a list of distinguished cycles, such that if two cycles in the list are contained in a theta graph, then the third cycle of the theta graph must also be in the list [25]. Recently, Slilaty has developed an orientation scheme for certain biased graphs and their matroids [20], answering a general orientation question posed by Zaslavsky [27].

A *gain graph* is a graph with the additional structure that each orientation of an edge is given a group element, called a *gain*, which is the inverse of the group element assigned to the opposite orientation. More recently, Slilaty also worked on real gain graphs and their orientations [19], generalizing graphic hyperplane arrangements.

In this paper, we define a notion of orientation for gain graphs over an arbitrary group. The orientation provides two immediate applications. First, a natural method for studying line graphs of gain graphs. And second, a well defined incidence matrix. In particular, gain graphs with complex unit gains (also called *complex unit gain graphs*) have particularly nice matrix and eigenvalue properties, which were initially investigated in [17].

The approach here fits the original orientation methods of Zaslavsky on signed graphs, and opens possibilities for future research. A major question left to answer is what an acyclic orientation is under this setup. This could lead to further connections with hyperplane arrangements and matroids. Additional questions for future projects are also posed throughout the paper.

The reader may also be interested in other recent independent investigations of complex unit gain graphs and specializations that appear in the literature. These include *weighted directed graphs* [1,13,12,11,14], which consider gains from the fourth roots of unity instead of the entire unit circle, and so called *Hermitian graphs* used to study universal state transfer [5]. A study of the characteristic polynomial for gain graphs has also been conducted in [10].

The paper is organized as follows. In Section 2, a background on the theory of gain graphs is provided. In Section 3, we develop oriented gain graphs. The construction introduced borrows from Edmonds and Johnson's definition of a bidirected graph [8] and Zaslavsky's more general oriented signed graphs [26]. Using this construction the line graph of an oriented gain graph is defined and studied in Section 4. If the gain group is abelian, the line graph of an oriented gain graph is used to define the line graph of a gain graph. This generalizes Zaslavsky's definition of the line graph of a signed graph [28].

Finally, in Section 5 we discuss several matrices associated to these various graphs and line graphs. For complex unit gain graphs, we generalize the classical relationship known for graphs and signed graphs between the incidence matrix of a graph and the adjacency matrix of the line graph. This is used to study the adjacency eigenvalues of the line graph.