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The characteristic subspace lattice of a linear transformation

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ABSTRACT

Given a square matrix $A \in M_n(\mathbb{F})$, the lattices of the hyperinvariant (Hinv(A)) and characteristic (Chinv(A)) subspaces coincide whenever $\mathbb{F} \neq GF(2)$. If the characteristic polynomial of A splits over $\mathbb{F}$, A can be considered nilpotent. In this paper we investigate the properties of the lattice Chinv(J) when $\mathbb{F} = GF(2)$ for a nilpotent matrix J. In particular, we prove it to be self-dual.

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1. Introduction

Let $\mathbb{F}^n$ be the n-dimensional vector space over a field $\mathbb{F}$, and $A \in M_n(\mathbb{F})$ a square matrix corresponding to an endomorphism of $\mathbb{F}^n$ in a fixed basis. A vector subspace $V \subseteq \mathbb{F}^n$ is called invariant with respect to the endomorphism if $AV \subseteq V$. The subspace V is hyperinvariant if it is invariant for every matrix $T \in Z(A)$ (i.e. commuting with A). Weakening the latter condition, if it is only satisfied for every nonsingular matrix T commuting with A, the subspace is called characteristic. Obviously

$$\text{Hinv}(A) \subseteq \text{Chinv}(A) \subseteq \text{Inv}(A),$$

where $\text{Hinv}(A)$, $\text{Chinv}(A)$ and $\text{Inv}(A)$ denote the lattices of hyperinvariant, characteristic and invariant subspaces, respectively.

For an arbitrary field $\mathbb{F}$, the lattice $\text{Inv}(A)$ is studied in [3], where it is proven to be self-dual, and characterizations of some other properties are given, for instance when it is distributive or Boolean, among others. A full description of $\text{Hinv}(A)$ when $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ can be found in [5], where it is proven to be a distributive and self-dual lattice, and tight bounds for its cardinality are provided. Concerning $\text{Chinv}(A)$, if the characteristic polynomial of A splits over $\mathbb{F}$ and $\text{card}(\mathbb{F}) > 2$, $\text{Chinv}(A) = \text{Hinv}(A)$ [1]. When $\text{card}(\mathbb{F}) = 2$, $\text{Chinv}(A)$ and $\text{Hinv}(A)$ in general do not coincide. Moreover, if all of the eigenvalues of A are in $\mathbb{F}$, the study of $\text{Hinv}(A)$ and $\text{Chinv}(A)$ can be reduced to the case where A has a unique eigenvalue (see, for instance [1,2,5]). Therefore, if the characteristic polynomial of A splits over $\mathbb{F}$, we can assume A to be a nilpotent matrix.

If A is a nilpotent matrix, and $\text{card}(\mathbb{F}) = 2$, Shoda's Theorem (see for instance [2]) characterizes the existence of characteristic non-hyperinvariant subspaces. General conditions for their existence, as well as some examples, can be found in [1,2]. A construction to explicitly obtain all of the characteristic non-hyperinvariant subspaces of A is given in [7].

Our aim in this paper is to analyze basic properties of the lattice of the characteristic subspaces $\text{Chinv}(A)$ of a nilpotent matrix A when $\mathbb{F} = GF(2)$. In particular we will prove that it is a self-dual lattice.

The paper is organized as follows: In section 2 we introduce the notation and basic results. We present here the structure of the characteristic non-hyperinvariant subspaces of A as obtained in [7]. In section 3 we analyze the properties of the lattice $\text{Chinv}(A)$. In particular, we give an anti-isomorphism from $\text{Chinv}(A)$ to $\text{Chinv}(A)$, hence proving that the lattice is self-dual.

2. Preliminaries

Throughout the paper we will assume that $\mathbb{F} = GF(2)$ and $A = J$ a nilpotent Jordan matrix. Given a set of vectors $\{v_1, \ldots, v_t\} \subset \mathbb{F}^n$, we represent by $\text{span}\{v_1, \ldots, v_t\}$ the vector subspace of linear combinations of the vectors $\{v_1, \ldots, v_t\}$. If E, F are vector subspaces of $\mathbb{F}^n$, the notation $E \cong F$ means that they are isomorphic.