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Spectral properties of small Hadamard matrices

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ABSTRACT

We prove that if A and B are Hadamard matrices which are both of size 4×4 or 5×5 and in dephased form, then $tr(A) = tr(B)$ implies that A and B have the same eigenvalues, including multiplicity. We calculate explicitly the spectrum for these matrices. We also extend these results to larger Hadamard matrices which are permutations of the Fourier matrix and calculate their spectral multiplicities.

1. Introduction

The matrix of the discrete Fourier transform for the finite group $\mathbb{Z}_n$, where $n \geq 2$ is an integer, is

$$\mathcal{F}_n := \frac{1}{\sqrt{n}} \left(e^{\frac{2\pi i jk}{n}} \right)_{j,k}.$$

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The problem of finding the eigenvalues and their multiplicity for the discrete Fourier transform has a long history (see, e.g. [1]) and has connections to many areas such as harmonic analysis, number theory, and numerical analysis. For example, one can see immediately that the trace of the matrix of the discrete Fourier transform is the Gauss sum

$$\sum_{j=0}^{n-1} e^{\frac{2\pi i j^2}{n}}.$$

It took Gauss several years to give a complete formula for these sums. And it can be used to give a short proof for the famous quadratic reciprocity law, which can be expressed in the following formula:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \frac{\mathrm{Tr}(\mathcal{F}_{pq})}{\mathrm{Tr}(\mathcal{F}_p)\mathrm{Tr}(\mathcal{F}_q)}.$$

The eigenvalues and eigenvectors for the Fourier matrix were computed (without the use of Gauss sums!) by McClelland and Parks in the paper [9] which appeared in IEEE Transactions on Audio and Electroacoustics. A simpler computation of the multiplicities of the eigenvalues is possible with the use of Gauss sums (see [1]). Thus, the spectrum of the Fourier matrix unveils some deep results in number theory. In this paper we calculate the spectrum of several classes of Hadamard matrices.

Definition 1. An $n \times n$ matrix H is called a *Hadamard matrix* if it is unitary, i.e., $HH^* = H^*H = I_n$, and all the entries have complex modulus $\frac{1}{\sqrt{n}}$. In some contexts, such an H is called a complex Hadamard matrix, but we will not make a distinction between real and complex H. Also, in most contexts, a Hadamard matrix is not normalized to be unitary, and has entries of complex modulus 1.

Hadamard matrices appear in a number of contexts. See [10] for a history of their development and many examples of their uses, including computing and quantum physics. Recent applications to quantum permutation groups are discussed in [2]. For example, a correspondence is shown between symmetries of a Hadamard matrix and quantum permutation groups. We are motivated by the appearance of Hadamard matrices in the context of the Fuglede conjecture [5,4] and Fourier analysis on fractals [8].

Every Hadamard matrix H can be factored as $H = D_1 H_0 D_2$ where D_1 and D_2 are diagonal matrices with diagonal entries of modulus 1, and H_0 is again a Hadamard matrix, but in *dephased form*, that is, all the entries in the first row and first column are $\frac{1}{\sqrt{n}}$. Henceforth, we will assume that a Hadamard matrix is in dephased form.

We denote the entries of a matrix M by $M[j,k]$. For convenience, we will index the rows and columns of an $n \times n$ matrix by $\{0, 1, \ldots, n-1\}$. For a Hadamard matrix H in dephased form, the 0th row and column have entries which are all $\frac{1}{\sqrt{n}}$.