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Boolean convolution in the quaternionic setting

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ABSTRACT

In this paper we begin a study of free analysis in the quaternionic setting, and consider Boolean convolution for quaternion-valued measures. To this end we also study Boolean convolution for matrix-valued complex measures, also proving Boolean infinite divisibility and central limit theorems for these measures. Moreover we prove an integral representation for quaternionic Carathéodory and Herglotz functions.

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1. Introduction

In this paper we begin a study of free analysis in the quaternionic setting. We denote by $\mathbb{H}$ the skew-field of quaternions and by $\mathbb{S}$ the sphere of purely imaginary quaternions q such that $q^2 = -1$ (basic definitions on quaternions and hyperholomorphic functions are recalled in Section 3.1). Let i be any fixed element in $\mathbb{S}$. Then it is possible to choose $j \in \mathbb{S}$ orthogonal to i, and any quaternion $q \in \mathbb{H}$ can be written as $q = x_0 + x_1 i + x_2 j + x_3 ij = z_1 + z_2 j$ where $x_\ell \in \mathbb{R}$, $\ell = 0, \ldots, 3$ and z_1, z_2 belong to the complex plane whose imaginary unit is i (and, in particular, z_1, z_2 can be real numbers).

Motivated by the work [3], where the notion of quaternionic q-positive measure on $[0, 2\pi]$ was introduced, we consider in this paper measures $dn(t) = dn_0(t) + j dn_1(t)$ on $i\mathbb{R}$ identified with the real line, with real-valued components dn_0 and dn_1 with the following properties: The measure dn_0 is positive and even, dn_1 is a signed odd measure, and the $\mathbb{R}^{2\times 2}$-valued measure

$$d\widetilde{n}(t) = \begin{pmatrix} dn_0(t) & dn_1(t) \\ dn_1(t) & dn_0(t) \end{pmatrix}$$

is positive. Such measures will be called j-positive. For a fixed choice of (i, j), these measures are in one-to-one correspondence with slice-hyperholomorphic functions mapping the right half-plane $\mathbb{H}_+$ into itself, with a growth condition at infinity. As is well known (see e.g. [15]), such functions in the complex setting (and with $\mathbb{H}_+$ replaced by the open upper half-plane) play a key role in free analysis.

In order to study the quaternionic case, it is sometimes useful to consider, instead of quaternionic-valued functions, $\mathbb{C}^{2\times 2}$-valued functions. This fact naturally leads us to consider the more general case of $\mathbb{C}^{n\times n}$-valued Herglotz functions. Moreover we studied the moments and the Boolean convolution of two $\mathbb{C}^{n\times n}$-valued measures and related problems (see [5] for related work). Then we show that a normalized $\mathbb{C}^{n\times n}$-valued measure with compact support is Boolean infinitely divisible. We also prove a central limit theorem.

We note that among the 2×2 matrices useful in the quaternionic setting appears the algebra $\mathcal{A}$ of $\mathbb{R}^{2\times 2}$-valued matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a, b \in \mathbb{R}$. These are the hyperbolic numbers from hypercomplex analysis (see [4] for the latter). A key fact in the arguments is that the moments (when defined) of a j-positive measure belong to $\mathcal{A}$. Since sums and products of moments stay in the algebra $\mathcal{A}$, we define the Boolean convolution of two j-positive measures μ and ν via the convolutions of the associated measures $\widetilde{\mu}$ and $\widetilde{\nu}$.

In summary, the key idea is to first study Boolean (additive) convolution for complex matrix-valued normalized (that is, with full mass equal to I_n) measures with compact support. Then we move to a subfamily of these measures, called j-positive (see also [2]), whose moments belong to the above mentioned algebra $\mathcal{A}$. This is a specific feature of the quaternionic case, which is needed to develop the corresponding theory. As a