Linear Algebra and its Applications 506 (2016) 445–477

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# Ultra discrete permanent and the consistency of max plus linear equations

N. Shinzawa

*Nishinippon Institute of Technology, 1-11, Aratsu, Kanda-machi, Miyako-gun, Fukuoka, 800-0394, Japan*

ARTICLE INFO

*Article history:*
Received 14 February 2015
Accepted 18 May 2016
Available online 24 May 2016
Submitted by P. Butkovic

*MSC:*
15A15

*Keywords:*
Max plus algebra
Tropical algebra
Integrable system

ABSTRACT

In this paper, we investigate the consistency conditions for three classes of the max plus linear equations, including the case corresponding to the convex polyhedra which was the subject of the previous paper. The necessary and sufficient conditions for the existence of finite solutions are expressed in the form of single equations, by using the ultra discrete permanent.



## 1. Introduction

The determinant of a matrix is one of the most important concept of the classical linear algebra. The max plus algebraic counter part [1] of the determinant is obtained by replacing the multiplication with addition and replacing both of the addition and the substraction with the max operator.

$$|A| = \max_{i_1,\ i_2,\ \cdots,\ i_n \in S_n} (a_{1i_1} + a_{2i_2} + \cdots + a_{ni_n})$$

*E-mail address:* shinzawa@nishitech.ac.jp.

http://dx.doi.org/10.1016/j.laa.2016.05.019

Here, $A$ is a $n \times n$ matrix and $a_{ij}$, $(i, j = 1,\ 2,\ \cdots,\ n)$ is the $ij$ element of $A$. The indices $i_1,\ i_2,\ \cdots,\ i_n$ take values of $1,\ 2,\ \cdots,\ n$ and the maximum is taken over all permutations of them. This counterpart of the determinant is called as the ultra discrete permanent [5,6] in the context of the study of the integrable system [9–11], and is also called as the tropical determinant [2].

One of the origins of the determinant is the study of the consistency conditions for the linear equations. There are several researches concerning to this subject. In the book [3], section 2, a large class of the max plus linear equations, which include the equations (1) and (3) below, is investigated. The condition for the max plus linear equations to have a solution was stated from the graph theoretical point of view. The uniqueness of the solution was also stated in the same context. In the paper [2], the condition for the general autonomous max plus linear equations to have non-trivial solutions was studied in the symmetrization of the max plus algebra. The result was written down by using the determinant of the symmetrized algebra (see also [3]). In the book [4], the general max plus linear equation was studied and an algorithm to determine the consistency is proposed.

In the meantime, the cell automaton model which has exact N soliton solution was found in these decades. These models are now known to be obtained by taking a special limit of the integrable equations, and are expressed as difference equations which contain the max operator and the summation instead of the summation and the multiplication. Despite this correspondence, some of the properties of the integrable equations do not hold for soliton cell automata. The Bäcklund transformation, which is the transformation between the solutions of the integrable equations, was not found. To construct the Bäcklund transformation, we need to investigate the condition that the max plus linear equations possess the finite real solutions. In the previous paper, the author study such kind of the consistency condition, and obtain the following results.

(i) The consistency condition of the general two max plus linear equations for two variables can be expressed in the form of equations of the ultra discrete permanent which correspond to the vanishing of the determinant [7,8].
(ii) The consistency condition of the max plus linear equations which correspond to the convex polyhedra (see the equations (1) below) can be expressed as a single equation of the ultra discrete permanent [8].

However, the consistency condition for the general max plus linear equations was not yet obtained.

In this paper, we study the consistency condition of the following three cases of the max plus linear equations.

Case 1

$$a_{ii} + x_i = \max_{k=1,\ \cdots,\ n} (a_{ik} + x_k),\ (i = 1,\ \cdots,\ n) \tag{1}$$