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Isotonic linear operators on the space of all convergent real sequences

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A B S T R A C T

In this work, we consider a natural preorder on $\mathfrak{c}$, the Banach space of all convergent real sequences, which is called convex majorization. We find a large class of bounded linear operators $T : \mathfrak{c} \to \mathfrak{c}$, which preserve convex majorization and denote this class by $\mathcal{I}$. Then some interesting properties of each $T \in \mathcal{I}$ are obtained.

1. Introduction and preliminaries

For $x, y \in \mathbb{R}^n$, the vector $x = (x_1, \ldots, x_n)$ is said to be majorized by $y = (y_1, \ldots, y_n)$, denoted by $x \prec y$, if

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$$\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow} \quad \text{and} \quad \sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for } k = 1, \ldots, n-1,$$

where $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ is the decreasing order of the components of x.

The notion of majorization has been extended by numerous mathematicians [2,3,5,8,9]. There are several equivalent conditions of majorization on $\mathbb{R}^n$. Hardy, Littlewood, and Polya in [4] proved that $(x_1, \ldots, x_n) = x \prec y = (y_1, \ldots, y_n)$ is equivalent to

$$\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i),$$

for all continuous convex functions $\phi : \mathbb{R} \to \mathbb{R}$. In fact, the previous characterization shows that if $x \prec y$, then $\mathrm{co}(x) \subseteq \mathrm{co}(y)$, where $\mathrm{co}(x)$ is the convex hull of the set of the components of x.

The topic of linear preservers is of interest to a large group of matrix theorists. For a survey of linear preserver problems see [10]. In 1989 [1], Ando characterized the linear operators which preserve majorization on $\mathbb{R}^n$. On the basis of row-stochastic matrices, Khalooei, Radjabalipour, and Salemi [6,7] introduced the concept of left matrix majorization and determined all linear operator preserving left matrix majorization. It is easy to see that for each $x, y \in \mathbb{R}^n$, the vector x is left matrix majorized by y if and only if

$$\mathrm{co}(x) \subseteq \mathrm{co}(y). \tag{1}$$

Let $\mathfrak{c}$ be the Banach space of all convergent real sequences with the supremum norm $\|x\| = \sup_{j \in \mathbb{N}} |x_j|$, for $x = (x_1, x_2, \ldots) \in \mathfrak{c}$. Also, $\mathfrak{c}_0$ is the Banach space of all real sequences which tend to zero. For abbreviation, we write $\mathrm{co}(x)$, instead of the convex combination of the set $\mathrm{Im}(x) = \{x_j : j \in \mathbb{N}\}$, where $x \in \mathfrak{c}$. Also, we use the notation $\lim x$, instead of $\lim_{j \to \infty} x_j$, for all $x \in \mathfrak{c}$. An element $x \in \mathfrak{c}$ can be represented by $\sum_{j \in \mathbb{N}} x_j \boldsymbol{e}_j$, where $\boldsymbol{e}_j : \mathbb{N} \to \mathbb{R}$ is defined by $\boldsymbol{e}_j(i) = \delta_{ij}$, the Kronecker delta.

We organize this paper as follows. In the next section, by using (1), we extend the notion of the left matrix majorization to discrete space $\mathfrak{c}$. Then some properties of bounded linear operators preserving such majorization are proved and some relevant examples are considered. Finally, we propose some related open problems to the interested readers.

2. Main results

We first define a preorder on $\mathfrak{c}$, which is said to be "convex majorization" as the following.