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Convexity and matrix means

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A B S T R A C T

In this article we present some mean inequalities for convex functions that lead to some generalized inequalities treating the arithmetic, geometric and harmonic means for numbers and matrices. Our first main inequality will be

$$\left(\frac{\nu}{\tau}\right)^{\lambda} \le \frac{((1-\nu)f(0)+\nu f(1))^{\lambda} - f^{\lambda}(\nu)}{((1-\tau)f(0)+\tau f(1))^{\lambda} - f^{\lambda}(\tau)} \le \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for the convex function f, when $\lambda \ge 1$ and $0 < \nu \le \tau < 1$. Moreover, when $\lambda = 1$, the inequality will be valid for operator convex functions.

Then by selecting an appropriate convex function, we obtain certain matrix inequalities. In particular, we obtain several mixed mean inequalities for operators using real and operator convexity. Our discussion will lead to new multiplicative refinements and reverses of the Heinz and Hölder inequalities for matrices, new and refined trace and determinant inequalities. The significance of this work is its general treatment, where convexity is the only needed property.

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1. Introduction

Let $\mathbb{M}_n$ be the algebra of $n \times n$ complex matrices, $\mathbb{M}_n^+$ be the cone of positive semidefinite matrices in $\mathbb{M}_n$ and $\mathbb{M}_n^{++}$ be the cone of strictly positive matrices in $\mathbb{M}_n$.

For two Hermitian matrices A and B in $\mathbb{M}_n$, we say $A \le B$ when $B - A \in \mathbb{M}_n^+$. For $A, B \in \mathbb{M}_n^{++}$, the operator arithmetic, geometric and harmonic means are defined, respectively, by

$$A\nabla_\nu B = (1-\nu)A + \nu B, A\#_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}}, A!_\nu B = \left(A^{-1}\nabla_\nu B^{-1}\right)^{-1},$$

where $0 \le \nu \le 1$. When $\nu = \frac{1}{2}$, we drop the ν from the above notations.

It is well known that $A!_\nu B \le A\#_\nu B \le A\nabla_\nu B$, $0 \le \nu \le 1$, for $A, B \in \mathbb{M}_n^{++}$.

One goal of this article is to present several relations between these means, generalizing and refining some inequalities in the literature. To obtain these operator inequalities, we mainly use two techniques. The first is to prove the corresponding numerical inequalities, then we apply a standard functional calculus idea. However, for some inequalities, we will be using operator convex and operator monotone functions. Recall that a real valued function $f : \mathbb{I} \to \mathbb{R}$ is called operator convex if $f((1-\nu)A + \nu B) \le (1-\nu)f(A) + \nu f(B)$, for $0 \le \nu \le 1$ and the Hermitian matrices $A, B \in \mathbb{M}_n$ with spectrum in the interval $\mathbb{I}$, and it is operator monotone increasing if $f(A) \le f(B)$ when $A \le B$. In this context, $f(A)$ is defined by $Uf(D)U^*$, where $A = UDU^*$ is the spectral decomposition of A. That is, U is unitary and D is the diagonal matrix whose diagonal entries are the eigenvalues of A.

Speaking of the eigenvalues of a Hermitian matrix A, we use the notation $\lambda_j(A)$ to mean the j-th eigenvalue of A, when written in a decreasing order.

The comparison between matrices has been done in different ways, among which the Löwener partial order $\le$ is the strongest. More precisely, when A and B are Hermitian such that $A \le B$, we infer that $\lambda_j(A) \le \lambda_j(B), \forall j$, which is another perspective to compare between A and B. Notice that the relation $\lambda_j(A) \le \lambda_j(B), \forall j$ implies $\sum_{j=1}^k \lambda_j(A) \le \sum_{j=1}^k \lambda_j(B)$ for $1 \le k \le n$. This last comparison is what we call weak majorization, and is denoted by $\prec_w$. Thus, we have

$$A \le B \Rightarrow \lambda_j(A) \le \lambda_j(B), \forall j \Rightarrow A \prec_w B \Rightarrow |||A||| \le |||B|||,$$

where $|||\ \ |||$ stands for unitarily invariant norms. Recall that these are norms on $\mathbb{M}_n$ that absorb unitary matrices. That is, $|||UAV||| = |||A|||$ when U and V are unitary. A useful example of unitarily invariant norms is the Hilbert–Schmidt or Frobenious norm $\|\ \ \|_2$ defined by

$$\|A\|_2 = \operatorname{tr}(AA^*) = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2},$$

where tr is the trace functional.