Linear Algebra and its Applications 503 (2016) 26–47

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Pseudo-inverses of difference matrices and their application to sparse signal approximation

Gerlind Plonka [a,*], Sebastian Hoffmann [b], Joachim Weickert [b]

[a] *Institute for Numerical and Applied Mathematics, University of Göttingen, Lotzestr. 16-18, 37083 Göttingen, Germany*

[b] *Mathematical Image Analysis Group, Faculty of Mathematics and Computer Science, Campus E1.7, Saarland University, 66041 Saarbrücken, Germany*

ARTICLE INFO

Article history:
Received 16 April 2015
Accepted 20 March 2016
Available online 4 April 2016
Submitted by V. Mehrmann

MSC:
15A09
35K35
41A15
41A29
94A12

Keywords:
Moore–Penrose inverse
Fourier transform
Linear diffusion
Orthogonal matching pursuit
Partial differential equations
Interpolation

ABSTRACT

We derive new explicit expressions for the components of Moore–Penrose inverses of symmetric difference matrices. These generalized inverses are applied in a new regularization approach for scattered data interpolation based on partial differential equations. The columns of the Moore–Penrose inverse then serve as elements of a dictionary that allow a sparse signal approximation. In order to find a set of suitable data points for signal representation we apply the orthogonal matching pursuit (OMP) method.

* Corresponding author.
E-mail addresses: plonka@math.uni-goettingen.de (G. Plonka), hoffmann@mia.uni-saarland.de (S. Hoffmann), weickert@mia.uni-saarland.de (J. Weickert).

http://dx.doi.org/10.1016/j.laa.2016.03.033

1. Introduction

Within the last years, there have been several attempts to derive new sparse representations for signals and images in the context of sparse data interpolation with partial differential equations. The essential key of these so-called inpainting methods is to fix a suitable set of data points that admits a very good signal or image approximation, when the unspecified data are interpolated by means of a diffusion process. Finding optimal sparse data leads to a challenging optimization problem that is NP-hard, and different heuristic strategies have been proposed. Using nonlinear anisotropic diffusion processes, this approach has been introduced in [5] and strongly improved since that time by more sophisticated optimization methods such as [18]. Inpainting based on linear differential operators such as the Laplacian is conceptually simpler, but can still provide sparse signal approximations with high quality; see e.g. [6,14,15]. These recent results outperform earlier attempts in this direction that use specific features such as edges [3,9,4] or toppoints in scale-space [10].

In this paper, we focus on the one-dimensional discrete case. The goal is to provide new insights into this problem from a linear algebra perspective. In the first part we derive a formula for the Moore–Penrose inverse of difference matrices. This formula has already been proposed for finding generalized inverses of circulant matrices of size $N \times N$ with rank $N-1$ in [20] and for discrete Laplace operators considered e.g. in [22,17]. We generalize this formula to arbitrary symmetric $N \times N$ difference matrices of rank $N-1$ possessing the eigenvector $\mathbf{1} = (1, \ldots, 1)^T$ to the single eigenvalue 0. This formula enables us to derive explicit expressions for the components of the Moore–Penrose inverse for special difference matrices that play a key role in discrete diffusion inpainting. In particular, we study difference matrices that approximate the second and fourth order signal derivative, i.e., the one-dimensional Laplace operator and the biharmonic operator, with periodic resp. reflecting (homogeneous Neumann) boundary conditions.

With the help of the generalized inverse of the difference matrices, we propose a new regularization approach that admits different optimization strategies for finding a suitable sparse set of data points for signal reconstruction. In particular, we employ the orthogonal matching pursuit (OMP) method as a conceptually simple and efficient greedy algorithm to construct the desired set of data points.

The paper is organized as follows. In Section 2 we present the formula for the Moore–Penrose inverse of symmetric difference matrices and derive explicit expressions for the components of the Moore–Penrose inverse of difference matrices corresponding to the discrete Laplace and biharmonic operator using the theory of difference equations. Section 3 is devoted to a new regularization approach to the discrete inpainting problem. The columns of the explicitly given Moore–Penrose inverses of the difference matrices can be understood as discrete Green's functions [8] and are used as a dictionary for the orthogonal matching pursuit algorithm (OMP). Since the original discrete inpainting problem (with the Laplace or biharmonic operator) is equivalent to a least squares spline approximation problem with free knots, our approach can also be seen as an alter-