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On the third largest eigenvalue of graphs

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ABSTRACT

Let G be a graph with eigenvalues $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. In this paper we investigate the value of $\lambda_3(G)$. We show that if the multiplicity of -1 as an eigenvalue of G is at most $n-13$, then $\lambda_3(G) \geq 0$. We prove that $\lambda_3(G) \in \{-\sqrt{2}, -1, \frac{1-\sqrt{5}}{2}\}$ or $-0.59 < \lambda_3(G) < -0.5$ or $\lambda_3(G) > -0.496$. We find that $\lambda_3(G) = -\sqrt{2}$ if and only if $G \cong P_3$ and $\lambda_3(G) = \frac{1-\sqrt{5}}{2}$ if and only if $G \cong P_4$, where P_n is the path on n vertices. In addition we characterize the graphs whose third largest eigenvalue equals -1. We find all graphs G with $-0.59 < \lambda_3(G) < -0.5$. Finally we investigate the limit points of the set $\{\lambda_3(G)$: G is a graph such that $\lambda_3(G) < 0\}$ and show that 0 and -0.5 are two limit points of this set.

1. Introduction

Throughout this paper all graphs are simple, that is finite and undirected without loops and multiple edges. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. The adjacency

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matrix of G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and otherwise $a_{ij} = 0$. Thus $A(G)$ is a symmetric matrix with zeros on the diagonal and all the eigenvalues of $A(G)$ are real. By the eigenvalues of G we mean those of its adjacency matrix. We denote the eigenvalues of G by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. By the spectrum of G which is denoted by $Spec(G)$, we mean the multiset of eigenvalues of G. The characteristic polynomial of G, $det(\lambda I - A(G))$, is denoted by $P(G, \lambda)$. One of the attractive problems is finding the location of eigenvalues of graphs. We know that for every graph G, $\lambda_1(G) \geq 0$ while for the second largest eigenvalue $\lambda_2(G) = -1$ or $\lambda_2(G) \geq 0$ (in fact $\lambda_2(G) = -1$ if and only if G is complete graph, see [5]). So it is natural to find the location of the third largest eigenvalue. There are some papers related to this topic, see [1,2]. In [2] it is shown that there is no graph G such that $-1 < \lambda_3(G) < \frac{1-\sqrt{5}}{2}$. In this paper we obtain more results related to the value of third largest eigenvalue of graphs. We show that there is a relation between the multiplicity of -1 and the sign of λ_3. In fact we show that if G has at most $n - 13$ eigenvalues equal to -1, then $\lambda_3(G) \geq 0$. We prove that for every graph G,

$$\lambda_3(G) \in \{-\sqrt{2}, -1, \frac{1-\sqrt{5}}{2}\} \cup (-0.59, -0.5) \cup (-0.496, \infty).$$

For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively; $\overline{G}$ denotes the complement of G. The *order* of G denotes the number of vertices of G. The *closed neighborhood* of a vertex v of G which is denoted by $N[v]$, is the set $\{u \in V(G) : uv \in E(G)\} \cup \{v\}$. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$. We say G is k-regular if every vertex of G has degree k. For two graphs G and H with disjoint vertex sets, $G + H$ denotes the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$, is the disjoint union of G and H. In particular, nG denotes the disjoint union of n copies of G. The complete product (join) $G \vee H$ of graphs G and H is the graph obtained from $G + H$ by joining every vertex of G with every vertex of H. Let H be a subgraph of G. By $G \setminus H$ we mean the graph that is obtained from G by deleting the edges of H. For positive integers $n_1, \ldots, n_\ell$, $K_{n_1,\ldots,n_\ell}$ denotes the complete multipartite graph with ℓ parts of sizes $n_1, \ldots, n_\ell$. In particular $K_{a,b}$ is the complete bipartite graph with two parts of sizes a and b. Let K_n, $nK_1 = \overline{K_n}$, C_n and P_n denote the complete graph, the null graph, the cycle and the path on n vertices, respectively.

In this paper we find that $\lambda_3(G) = -\sqrt{2}$ if and only if $G \cong P_3$ and $\lambda_3(G) = \frac{1-\sqrt{5}}{2}$ if and only if $G \cong P_4$. In addition we prove that $\lambda_3(G) = -1$ if and only if $G \cong K_n$ or $G \cong K_p + K_q$ or $G \cong K_{a+b+c} \setminus K_{a,b}$, where n, p, q, a, b, and c are some positive integers such that $n \geq 3$ and $p + q \geq 3$ and $a + b + c \geq 4$. We characterize all graphs G with $-0.59 < \lambda_3(G) < -0.5$. Finally we investigate the limit points of the set $\{\lambda_3(G) :$ G is a graph such that $\lambda_3(G) < 0\}$ and show that 0 and -0.5 are two limit points of this set. We note that every rational eigenvalue of graphs is integer. Thus there is no graph G with $\lambda_3(G) = -0.5$.