Linear Algebra and its Applications 503 (2016) 180–189

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Proof of a problem on Laplacian eigenvalues of trees ☆

Xiying Yuan

Department of Mathematics, Shanghai University, Shanghai, 200444, China

A R T I C L E   I N F O

Article history:
Received 14 August 2015
Accepted 23 March 2016
Available online 11 April 2016
Submitted by R. Brualdi

MSC:
15A42
05C50

Keywords:
Tree
Perfect matching
k-th Laplacian eigenvalue

A B S T R A C T

Denote by $\mu_k(L(T))$ the k-th Laplacian eigenvalue of a tree T. Let $\mathcal{T}_k(2t)$ be the set of all trees of order $2tk$ with perfect matchings. In this note, the trees T in $\mathcal{T}_k(2t)$ with $\mu_k(L(T)) = \frac{t+2+\sqrt{t^2+4}}{2}$ are characterized, which solves Problem of J.X. Li, W.C. Shiu and A. Chang in [3] completely.

1. Introduction

Let G be a simple graph and $D(G)$ be the diagonal matrix of vertices degrees. The Laplacian matrix $L(G)$ of G is defined as $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G. Write $|G|$ for the order of graph G. For an edge e of G, denote by $G - e$ the subgraph of G obtained by deleting the edge e; for a vertex v of G, denote by $G - v$ the subgraph of G obtained by deleting the vertex v and all the edges incident to v.

☆ Research supported by NSFC (No. 11101263), and by a grant of "The First-class Discipline of Universities in Shanghai".

E-mail address: xiyingyuan2007@hotmail.com.

http://dx.doi.org/10.1016/j.laa.2016.03.040

For a real symmetric matrix M, the characteristic polynomial $det(xI-M)$ is denoted by $\Phi(M;x)$, and denote $\mu_k(M)$ the k-th eigenvalue of M (of order n), and the inequalities

$$\mu_1(M) \geq \mu_2(M) \geq \cdots \geq \mu_n(M)$$

hold. Write

$$a(t) = \frac{t+2+\sqrt{t^2+4}}{2}$$

for short in this paper. Denote

$$\mathcal{T}_k(2t) = \Big\{T : \text{tree } T \text{ has a perfect matching with } |T| = 2tk\Big\}.$$

For a tree T in $\mathcal{T}_k(2t)$ the value $\mu_k(L\,(T))$ was studied by Guo et al. in [4] and Li et al. in [3].

Lemma 1. *(See [4].) When $t = 1$, we have $\mu_k(L(T)) = 2$ for any tree T in $\mathcal{T}_k(2t)$.*

Lemma 2. *(See [3].) When $t \geq 2$, we have $\mu_k(L(T)) \leq a(t)$ for any tree T in $\mathcal{T}_k(2t)$.*

Let T^* (it is written as $T(2t,t)$ in [3]) be the tree obtained from the star $K_{1,t-1}$ ($t \geq 2$) by adding a new pendent edge at each vertex. Write kT^* for the union of k disjoint T^*. Denote

$$\mathbb{T}_k(2t) = \Big\{T : \text{tree } T \text{ contains } kT^* \text{ as a spanning subgraph}\Big\}.$$

Clearly, $\mathbb{T}_k(2t) \subseteq \mathcal{T}_k(2t)$. In [3], an algebraic characterization in terms of $\mu_k(L(T))$ for a tree T in $\mathbb{T}_k(2t)$ was proposed as Problem 3.

Problem 3. (See [3].) Let T be a tree in $\mathcal{T}_k(2t)$ and $t \geq 2$. Then $\mu_k(L(T)) = a(t)$ if and only if T is a tree in $\mathbb{T}_k(2t)$.

Lemma 4. *(See [1].) Let G be a graph of order n obtained from G' by deleting an edge. Then the eigenvalues of $L(G')$ and $L(G)$ interlace, that is,*

$$\mu_1\left(L(G')\right) \geq \mu_1\left(L(G)\right) \geq \mu_2\left(L(G')\right) \geq \mu_2\left(L(G)\right) \geq \cdots \geq \mu_n\left(L(G')\right) \geq \mu_n\left(L(G)\right).$$

The proof for the sufficiency part of Problem 3 can rely on Remark of [3], and for completeness, we restate here. Since $\mu_1(L\,(T^*)) = a(t)$ (see Lemma 5), we have

$$\mu_1(L\,(kT^*)) = \mu_2(L\,(kT^*)) = \cdots = \mu_k(L\,(kT^*)) = a(t).$$

If T is a tree in $\mathbb{T}_k(2t)$, i.e., T contains kT^* as a spanning subgraph, from Lemmas 4 and 2 we have