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Matrices totally positive relative to a tree, II

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ABSTRACT

If T is a labelled tree, a matrix A is totally positive relative to T, principal submatrices of A associated with deletion of pendent vertices of T are P-matrices, and A has positive determinant, then the smallest absolute eigenvalue of A is positive with multiplicity 1 and its eigenvector is signed according to T. This conclusion has been incorrectly conjectured under weaker hypotheses.

1. Introduction

A real matrix is called *totally positive* (TP) if all its minors are positive, and it is a P-matrix if every principal minor is positive.

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In [1] the following weakening has been studied. An n-by-n real matrix is *totally positive relative to a given labelled tree* T on n vertices (T-TP) if, for each pair of pendent vertices p and q of T, the matrix $A[\alpha]$ is TP when α is the ordered set of vertices of the unique induced path of T that connects p and q. If T is a path with vertices labelled in order, then TP and T-TP are the same. Note that we are going to refer to T throughout as a labelled tree.

Of course, T-TP equivalently means that $A[\alpha]$ is TP for the vertices of any induced path of T, as the unique path joining any pair of vertices of T is a subpath of some path joining pendent vertices.

It is known that a totally positive matrix has distinct positive eigenvalues and that the smallest one has an eigenvector that alternates in sign (see [2] for general background). Since a tree is bipartite, there is a signing of the vertices so that neighbors have different signs. For a labelled tree, T, let σ be a ± 1 vector consistent with such a signing. We say that σ is *signed according to* T, and σ is unique up to multiplication by ± 1. It had been conjectured that if A is T-TP, then A has a unique absolute smallest real eigenvalue with an eigenvector signed according to T. We call this the *Neumaier conclusion*, after the original conjecture by Arnold Neumaier, University of Vienna. See [1] for prior work.

This conjecture was proven for a few trees, but is false in general. Here, our purpose is to prove the original conjecture for all trees by adding a hypothesis.

2. Notation and terminology

Let us denote the set $\{1, \ldots, n\}$ by N; Moreover, we will denote by N_i (resp. $N_{i,j}$, and $N_{i,j,k}$) the set $N \setminus \{i\}$ (resp. $N \setminus \{i, j\}$, and $N \setminus \{i, j, k\}$).

Let $A \in M_n(\mathbb{R})$. For any ordered index sets α, $\beta \subseteq N$, with $|\alpha| = |\beta| = k$, by $A[\alpha; \beta]$ we mean the k-by-k submatrix of A that lies in the rows indexed by α and the columns indexed by β, and with the order of the rows (resp. columns) determined by the order in α (resp. β), by $A[\alpha]$ we mean $A[\alpha; \alpha]$, by $A(i; j)$ we mean the $(n-1)$-by-$(n-1)$ submatrix of A that lies in the rows indexed by N_i and the columns indexed by N_j, and by $A(i)$ we mean $A(i; i)$.

Suppose that T is a labelled tree on n vertices. If $\mathscr{P}$ is an induced path of T, by $A[\mathscr{P}]$ we mean $A[\alpha]$ in which α consists of the indices of the vertices of $\mathscr{P}$ in the order in which they appear along $\mathscr{P}$. Since everything we discuss is independent of reversal of order, there is no ambiguity regarding intended direction.

Definition 1. For a given labelled tree T on n vertices, we say that $A \in M_n(\mathbb{R})$ is T-TP if $A[\mathscr{P}]$ is TP for each path $\mathscr{P}$ connecting any two pendent vertices.

Observe that for a T-TP matrix, properly less is required than for a TP matrix; however, like TP matrices, T-TP matrices are entry-wise positive.