

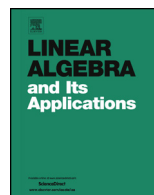


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## Matrices totally positive relative to a tree, II

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## ABSTRACT

If  $T$  is a labelled tree, a matrix  $A$  is totally positive relative to  $T$ , principal submatrices of  $A$  associated with deletion of pendent vertices of  $T$  are  $P$ -matrices, and  $A$  has positive determinant, then the smallest absolute eigenvalue of  $A$  is positive with multiplicity 1 and its eigenvector is signed according to  $T$ . This conclusion has been incorrectly conjectured under weaker hypotheses.

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## 1. Introduction

A real matrix is called *totally positive* (TP) if all its minors are positive, and it is a  $P$ -matrix if every principal minor is positive.

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In [1] the following weakening has been studied. An  $n$ -by- $n$  real matrix is *totally positive relative to a given labelled tree*  $T$  on  $n$  vertices (T-TP) if, for each pair of pendent vertices  $p$  and  $q$  of  $T$ , the matrix  $A[\alpha]$  is TP when  $\alpha$  is the ordered set of vertices of the unique induced path of  $T$  that connects  $p$  and  $q$ . If  $T$  is a path with vertices labelled in order, then TP and T-TP are the same. Note that we are going to refer to  $T$  throughout as a labelled tree.

Of course, T-TP equivalently means that  $A[\alpha]$  is TP for the vertices of any induced path of  $T$ , as the unique path joining any pair of vertices of  $T$  is a subpath of some path joining pendent vertices.

It is known that a totally positive matrix has distinct positive eigenvalues and that the smallest one has an eigenvector that alternates in sign (see [2] for general background). Since a tree is bipartite, there is a signing of the vertices so that neighbors have different signs. For a labelled tree,  $T$ , let  $\sigma$  be a  $\pm 1$  vector consistent with such a signing. We say that  $\sigma$  is *signed according to*  $T$ , and  $\sigma$  is unique up to multiplication by  $\pm 1$ . It had been conjectured that if  $A$  is T-TP, then  $A$  has a unique absolute smallest real eigenvalue with an eigenvector signed according to  $T$ . We call this the *Neumaier conclusion*, after the original conjecture by Arnold Neumaier, University of Vienna. See [1] for prior work.

This conjecture was proven for a few trees, but is false in general. Here, our purpose is to prove the original conjecture for all trees by adding a hypothesis.

## 2. Notation and terminology

Let us denote the set  $\{1, \dots, n\}$  by  $N$ ; Moreover, we will denote by  $N_i$  (resp.  $N_{i,j}$ , and  $N_{i,j,k}$ ) the set  $N \setminus \{i\}$  (resp.  $N \setminus \{i, j\}$ , and  $N \setminus \{i, j, k\}$ ).

Let  $A \in M_n(\mathbb{R})$ . For any ordered index sets  $\alpha, \beta \subseteq N$ , with  $|\alpha| = |\beta| = k$ , by  $A[\alpha; \beta]$  we mean the  $k$ -by- $k$  submatrix of  $A$  that lies in the rows indexed by  $\alpha$  and the columns indexed by  $\beta$ , and with the order of the rows (resp. columns) determined by the order in  $\alpha$  (resp.  $\beta$ ), by  $A[\alpha]$  we mean  $A[\alpha; \alpha]$ , by  $A(i; j)$  we mean the  $(n-1)$ -by- $(n-1)$  submatrix of  $A$  that lies in the rows indexed by  $N_i$  and the columns indexed by  $N_j$ , and by  $A(i)$  we mean  $A(i; i)$ .

Suppose that  $T$  is a labelled tree on  $n$  vertices. If  $\mathcal{P}$  is an induced path of  $T$ , by  $A[\mathcal{P}]$  we mean  $A[\alpha]$  in which  $\alpha$  consists of the indices of the vertices of  $\mathcal{P}$  in the order in which they appear along  $\mathcal{P}$ . Since everything we discuss is independent of reversal of order, there is no ambiguity regarding intended direction.

**Definition 1.** For a given labelled tree  $T$  on  $n$  vertices, we say that  $A \in M_n(\mathbb{R})$  is T-TP if  $A[\mathcal{P}]$  is TP for each path  $\mathcal{P}$  connecting any two pendent vertices.

Observe that for a T-TP matrix, properly less is required than for a TP matrix; however, like TP matrices, T-TP matrices are entry-wise positive.

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