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On nilpotent evolution algebras

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ABSTRACT

The type and several invariant subspaces related to the upper annihilating series of finite-dimensional nilpotent evolution algebras are introduced. These invariants can be easily computed from any natural basis. Some families of nilpotent evolution algebras, defined in terms of a nondegenerate, symmetric, bilinear form and some commuting, symmetric, diagonalizable endomorphisms relative to the form, are explicitly constructed. Both the invariants and these families are used to review and complete the classification of nilpotent evolution algebras up to dimension five over algebraically closed fields.

1. Introduction

Evolution algebras were introduced in 2006 by Tian and Vojtechovsky, in their paper "Mathematical concepts of evolution algebras in non-Mendelian genetics" (see [7]). Later on, Tian laid the foundations of evolution algebras in his monograph [8].

In some recent papers [4,5], a classification of the nilpotent evolution algebras up to dimension five has been given. However, there is a subtle point which has not been

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considered. When dealing with an extension of a nilpotent evolution algebra by a trivial ideal, one cannot fix a natural basis of the initial algebra, because a natural basis of a quotient does not necessarily extend to a natural basis of the whole algebra. As a consequence, the classifications in these papers are not complete. This also shows how tricky these algebras are.

The goal of this paper is the introduction of some new techniques for the study of evolution algebras, as well as the construction of several noteworthy families of nilpotent evolution algebras defined in terms of bilinear forms and symmetric endomorphisms. Using these tools, the classification of the nilpotent evolution algebras up to dimension five, over an algebraically closed field of characteristic not two, is obtained without much effort, although the number of possibilities in dimension five is quite high and indicates the difficulty of this problem for higher dimension.

Let us first recall the basic definitions.

An *evolution algebra* is an algebra $\mathcal{E}$ containing a countable basis (as a vector space) $B = \{e_1, \dots, e_n, \dots,\}$ such that $e_i e_j = 0$ for any $1 \leq i \neq j \leq n$. A basis with this property is called a *natural basis*. By its own definition, any evolution algebra is commutative. In this paper we deal with finite dimensional evolution algebras. Given a natural basis $B = \{e_1, \dots, e_n\}$ of an evolution algebra $\mathcal{E}$, $e_i^2 = \sum_{j=1}^{n} \alpha_{ij} e_j$ for some scalars $\alpha_{ij} \in \mathbb{F}$, $1 \leq i, j \leq n$. The matrix $A = \big(\alpha_{ij}\big)$ is the *matrix of structural constants* of the evolution algebra $\mathcal{E}$, relative to the natural basis B.

We recall next the definition of the graph and weighted graph attached to an evolution algebra (see [3, Definition 2.2]). Our graphs are always directed graphs, and most of our algebras will be presented by means of their graphs.

Let $\mathcal{E}$ be an evolution algebra with a natural basis $B = \{e_1, \dots, e_n\}$ and matrix of structural constants $A = \big(\alpha_{ij}\big)$.

- The graph $\Gamma(\mathcal{E}, B) = (V, E)$, with $V = \{1, \dots, n\}$ and $E = \{(i, j) \in V \times V : \alpha_{ij} \neq 0\}$, is called the *graph attached to the evolution algebra $\mathcal{E}$ relative to the natural basis B*.
- The triple $\Gamma^w(\mathcal{E}, B) = (V, E, \omega)$, with $\Gamma(\mathcal{E}, B) = (V, E)$ and where ω is the map $E \to \mathbb{F}$ given by $\omega\big((i, j)\big) = \alpha_{ij}$, is called the *weighted graph attached to the evolution algebra $\mathcal{E}$ relative to the natural basis B*.

The paper is organized as follows. In Section 2 we prove a general Krull–Schmidt Theorem for nonassociative algebras, which has its own independent interest. It shows that it is enough to classify indecomposable algebras. Also, using the annihilator of an algebra, we give some results which are useful to check the decomposability of a finite-dimensional algebra. See Lemma 2.4, Corollaries 2.5 and 2.6.

In Section 3 we define the upper annihilating series of an arbitrary nonassociative algebra, and then the type of a finite-dimensional nilpotent algebra. This allows us later on to split the classification of nilpotent evolution algebras according to their types.