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Hypercubes are determined by their distance spectra

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We show that the d-cube is determined by the spectrum of its distance matrix.

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1. Introduction

For undefined terms, see next section. For integers $n \geq 2$ and $d \geq 2$, the *Hamming graph* $H(d, n)$ has as vertex set, the d-tuples with elements from $\{0, 1, 2, \ldots, n-1\}$, with two d-tuples are adjacent if and only if they differ only in one coordinate. For a positive integer d, the *d-cube* is the graph $H(d, 2)$. In this paper, we show that the d-cube is determined by its distance spectrum, see Theorem 5.4. Observe that the d-cube has exactly three distinct distance eigenvalues.

Note that the d-cube is not determined by its adjacency spectrum as the Hoffman graph [11] has the same adjacency spectrum as the 4-cube and hence the cartesian product of $(d-4)$-cube and the Hoffman graph has the same adjacency spectrum as the d-cube (for $d \geq 4$, where the 0-cube is just K_1). This also implies that the complement of the d-cube is not determined by its distance spectrum, if $d \geq 4$.

There are quite a few papers on matrices of graphs with a few distinct eigenvalues. An important case when the Seidel matrix of a graph has exactly two distinct eigenvalues is very much studied, see, for example [18]. Van Dam and Haemers [20] characterized the graphs whose Laplacian matrix has two distinct nonzero eigenvalues. The non-regular graphs with three distinct adjacency eigenvalues studied in Bridges & Mena [4], Muzychuk & Klin [16], Van Dam [19], and Cheng et al. [6,7]. In this paper, we focus on connected graphs with three distinct distance eigenvalues.

Also the question whether a graph Γ is determined by its spectrum of a matrix $M = M(\Gamma)$, where M is the adjacency matrix, the Laplacian matrix, the signless Laplacian matrix and so on, has received much attention, see, for example, Van Dam and Haemers [21,22]. The study of the distance spectrum of a connected graph has obtained considerable attention in the last few years, see Aouchiche & Hansen [2], for a survey paper on this topic. The study of question whether a graph is determined by its distance spectrum is just in its beginning. See McKay [15], Lin et al. [14], and Jin & Zhang [12] for recent papers on this subject.

This paper is organized as follows. In Section 2, we will give definitions and preliminaries. In Section 3, we will give some old and new results on the second largest and smallest distance eigenvalue. We also will give an alternative proof for the fact that the complete multipartite graphs are determined by their distance spectrum, a result that was first shown by Jin & Zhang [12]. In Section 4, we will look at connected graphs with exactly three distinct distance eigenvalues and develop some basic theory for them. Proposition 4.3 is crucial for the proof of our main result. In Section 5, we will give a proof of the fact that the d-cube is determined by its distance spectrum, our main result. In Section 6, we will give some open problems.

2. Preliminaries

All the graphs in this paper are simple and undirected. A *graph* Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a finite set and $E(\Gamma) \subseteq \binom{V(\Gamma)}{2}$. The set $V(\Gamma)$ is the vertex