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More on regular subgroups of the affine group

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ABSTRACT

This paper is a new contribution to the study of regular subgroups of the affine group $\mathrm{AGL}_n(\mathbb{F})$, for any field $\mathbb{F}$. In particular we associate to each partition $\lambda \neq (1^{n+1})$ of $n+1$ abelian regular subgroups in such a way that different partitions define non-conjugate subgroups. Moreover, we classify the regular subgroups of certain natural types for $n \leq 4$. Our classification is equivalent to the classification of split local algebras of dimension $n+1$ over $\mathbb{F}$. Our methods, based on classical results of linear algebra, are computer free.

1. Introduction

Let $\mathbb{F}$ be any field. We identify the affine group $\mathrm{AGL}_n(\mathbb{F})$ with the subgroup of $\mathrm{GL}_{n+1}(\mathbb{F})$ consisting of the matrices having $(1, 0, \ldots, 0)^{\mathrm{T}}$ as first column. With this notation, $\mathrm{AGL}_n(\mathbb{F})$ acts on the right on the set $\mathcal{A} = \{(1, v) : v \in \mathbb{F}^n\}$ of affine points. Clearly, there exists an epimorphism $\pi : \mathrm{AGL}_n(\mathbb{F}) \to \mathrm{GL}_n(\mathbb{F})$ induced by the action of $\mathrm{AGL}_n(\mathbb{F})$ on $\mathbb{F}^n$. A subgroup (or a subset) R of $\mathrm{AGL}_n(\mathbb{F})$ is called *regular* if it acts

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regularly on $\mathcal{A}$, namely if, for every $v \in \mathbb{F}^n$, there exists a unique element in R having $(1, v)$ as first row. Thus R is regular precisely when $\mathrm{AGL}_n(\mathbb{F}) = \widehat{\mathrm{GL}}_n(\mathbb{F})\, R$, with $\widehat{\mathrm{GL}}_n(\mathbb{F}) \cap R = \{I_{n+1}\}$, where $\widehat{\mathrm{GL}}_n(\mathbb{F})$ denotes the stabilizer of $(1, 0, \ldots, 0)$.

A subgroup H of $\mathrm{AGL}_n(\mathbb{F})$ is *indecomposable* if there exists no decomposition of $\mathbb{F}^n$ as a direct sum of non-trivial $\pi(H)$-invariant subspaces. Clearly, to investigate the structure of regular subgroups, the indecomposable ones are the most relevant, since the other ones are direct products of regular subgroups in smaller dimensions (see beginning of Section 6). So, one has to expect very many regular subgroups when n is big. Actually, in Section 6 we show how to construct at least one abelian regular subgroup, called *standard*, for each partition $\lambda \neq (1^{n+1})$ of $n + 1$, in such a way that different partitions produce non-conjugate subgroups. Several of them are indecomposable.

The structure and the number of conjugacy classes of regular subgroups depend on $\mathbb{F}$. For instance, if $\mathbb{F}$ has characteristic $p > 0$, every regular subgroup is unipotent [14, Theorem 3.2], i.e., all its elements satisfy $(t - I_{n+1})^{n+1} = 0$. A unipotent group is conjugate to a subgroup of the group of upper unitriangular matrices (see [10]): in particular it has a non-trivial center. By contrast to the case $p > 0$, $\mathrm{AGL}_2(\mathbb{R})$ contains $2^{|\mathbb{R}|}$ conjugacy classes of regular subgroups with trivial center, hence not unipotent (see Example 2.5). So, clearly, a classification in full generality is not realistic.

Since the center $Z(R)$ of a regular subgroup R is unipotent (see Theorem 2.4(a)) if $Z(R)$ is non-trivial one may assume, up to conjugation, that R is contained in the centralizer of a unipotent Jordan form (see Theorem 4.4). But even this condition is weak.

Before introducing a stronger hypothesis, which allows to treat significant cases, we need some notation. We write every element r of R as

$$r = \begin{pmatrix} 1 & v \\ 0 & \pi(r) \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & \tau_R(v) \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & I_n + \delta_R(v) \end{pmatrix} = \mu_R(v), \tag{1}$$

considering the functions $\mu_R : \mathbb{F}^n \to \mathrm{AGL}_n(\mathbb{F})$, $\tau_R : \mathbb{F}^n \to \mathrm{GL}_n(\mathbb{F})$ and $\delta_R := \tau_R - \mathrm{id} : \mathbb{F}^n \to \mathrm{Mat}_n(\mathbb{F})$.

The hypothesis we introduce is that δ_R is linear. First of all, if R is abelian, then δ_R is linear (see [1]). Moreover, if δ_R is linear, then R is unipotent by Theorem 2.4(b), but not necessarily abelian. One further motivation for this hypothesis is that δ_R is linear if and only if $\mathcal{L} = \mathbb{F}I_{n+1} + R$ is a split local subalgebra of $\mathrm{Mat}_{n+1}(\mathbb{F})$. Moreover, two regular subgroups R_1 and R_2, with δ_{R_i} linear, are conjugate in $\mathrm{AGL}_n(\mathbb{F})$ if and only if the corresponding algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic (see Section 3). In particular, there is a bijection between conjugacy classes of abelian regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$ and isomorphism classes of abelian split local algebras of dimension $n + 1$ over $\mathbb{F}$. This fact was first observed by Caranti, Dalla Volta and Sala in [1]. It was studied also in connection with other algebraic structures in [3–6] and in [2], where the classification of nilpotent associative algebras given in [7] is relevant.

In Section 7 we classify, up to conjugation, certain types of regular subgroups U of $\mathrm{AGL}_n(\mathbb{F})$, for $n \leq 4$, and the corresponding algebras. More precisely, for $n = 1$