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Cauchy-like and Pellet-like results for polynomials

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We obtain several Cauchy-like and Pellet-like results for the zeros of a general complex polynomial by considering similarity transformations of the squared companion matrix and by treating the zeros of a scalar polynomial as the eigenvalues of a matrix polynomial.

1. Introduction

Two standard results for the localization of all or some of the zeros of a polynomial, due to Cauchy and Pellet, respectively, are given by

Theorem 1.1 *(Cauchy's theorem – original scalar version). ([3], [8, Th. (27,1), p. 122 and Exercise 1, p. 126]) All the zeros of the polynomial* $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$

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with complex coefficients, $n \geq 2$, lie in $|z| \leq s$, where s is the unique positive solution of

$$x^n - |a_{n-1}|x^{n-1} - \cdots - |a_1|x - |a_0| = 0 \ .$$

Theorem 1.2 *(Pellet's theorem – original scalar version). ([13], [8, Th. (28,1), p. 128]) Given the polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with complex coefficients, $a_k \neq 0$, and $n \geq 2$, let $1 \leq k \leq n-1$, and let the polynomial*

$$x^n + |a_{n-1}|x^{n-1} + \cdots + |a_{k+1}|x^{k+1} - |a_k|x^k + |a_{k-1}|x^{k-1} + \cdots + |a_1|x + |a_0|$$

have two distinct positive roots x_1 and x_2 with $x_1 < x_2$. Then p has exactly k zeros in or on the circle $|z| = x_1$ and no zeros in the annular ring $x_1 < |z| < x_2$.

Theorem 1.1 provides an upper bound on the moduli of the zeros, whereas Theorem 1.2 sometimes allows zeros to be separated into two different groups, according to the magnitude of their moduli. However, the latter is very sensitive to the magnitude of the coefficients and for the theorem to be applicable, one or more coefficients typically have to be much larger than the others.

The inequalities for the moduli of the zeros in these theorems are sharp in the sense that there exist polynomials for which they hold as equalities. Our aim is nevertheless to improve Theorem 1.1 and a few special cases of Theorem 1.2, resulting in a number of results of a similar nature, i.e., also involving the solution of one or two real equations. We immediately point out that the solution of such equations requires a negligible computational effort compared to the computation of the actual zeros (see, e.g., [12,15]), and we will not dwell on it. Theorems 1.1 and 1.2 have many applications and are often used to find good starting points for iterative methods that compute some or all of the zeros.

There are several ways to derive these and many other results related to polynomial zeros, one of which is to use linear algebra arguments. Although not necessarily producing the shortest proofs, it does provide a transparent and often elegant treatment of such results. On the other hand, a linear algebra approach does not generally seem to lead to results that cannot also be obtained by purely algebraic manipulation or applications of complex analysis, an observation also made in [14, p. 263]. Here, in contrast, we will use linear algebra tools to derive results that, it appears, cannot easily be obtained otherwise.

Before going into more detail, we recall that the zeros of the complex monic scalar polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ are the eigenvalues of the $n \times n$ *companion matrix* $C(p)$, defined by

$$C(p) = \begin{pmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix} .$$