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Completely positive rooted matrices

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The classes of k-rooted and ∞-rooted completely positive matrices are introduced. It is shown that completely positive matrices with at most two different eigenvalues are ∞-rooted, and completely positive singular ∞-rooted matrices of order 3 are characterized. Positive semidefinite matrices whose large powers are non-negative are also characterized.

1. Introduction

The class $\mathcal{CP}_n$ of completely positive matrices of order n consists of the matrices of the form BB^T, where B is an $n \times m$ real non-negative matrix; it is included in the class $\mathcal{DNN}_n$ of doubly non-negative matrices of order n, which consists of the symmetric entrywise non-negative real matrices with non-negative eigenvalues. This inclusion is an equality for $n \leq 4$, and there are examples showing that the inclusion is proper for every $n \geq 5$. We refer to the monograph by Berman and Shaked-Monderer [1] for more

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information on doubly non-negative and completely positive matrices; the interested reader is referred also to the more recent papers by Bomze, Shachinger and Uchida [3], Dickinson [4], Dür [5], Hiriart-Urruty and Seeger [8].

If $A \in \mathcal{DNN}_n$, then obviously $A^2 \in \mathcal{CP}_n$. Since every positive semidefinite matrix A admits a unique positive semidefinite square root, denoted by $\sqrt{A}$, one may ask when $A \in \mathcal{CP}_n$ implies that $\sqrt{A} \geq O$. These matrices deserve an *ad hoc* definition.

Definition 1.1. A doubly non-negative matrix A is called *rooted* if $\sqrt{A} \geq O$.

Obviously a rooted doubly non-negative matrix is completely positive. Thus, for $n \geq 5$, all matrices in $\mathcal{DNN}_n \setminus \mathcal{CP}_n$ are examples of non-rooted matrices. More interesting is to understand when a completely positive matrix is, or is not, rooted.

Doubly non-negative rooted matrices have already appeared in the literature. In fact, Marcus and Minc [10] proved that a doubly non-negative doubly stochastic $n \times n$ matrix ($n \geq 3$) with all diagonal entries $\leq 1/(n-1)$ is rooted (see Theorem 2.8, p. 127 in [1]); they also provided an example of a non-rooted doubly non-negative doubly stochastic matrix of order 3 (see Exercise 2.50, p. 129 in [1]). We will elaborate on that example in Section 4.

More recently, Shaked-Monderer proved (see the proof of Observation 1.1 in [13]) that non-negative matrices generated by Soules matrices are ∞-rooted (we refer to [15] and [6] for the definition of Soules matrices and the description of their structure). Furthermore, Reams [12] provided sufficient conditions on the diagonal elements of irreducible matrices, and on principal 2×2 minors of irreducible doubly stochastic matrices in $\mathcal{DNN}_n$, ensuring that they are rooted; these conditions are related to eigenvalues and eigenvectors of the matrices.

Of course, we can repeat the square root process, obtaining the following notion.

Definition 1.2. For a fixed positive integer k, a doubly non-negative matrix A is called *k-rooted* if $A^{1/2^k} \geq O$.

So, in this terminology, the rooted matrices are exactly the 1-rooted ones. The class of k-rooted matrices of order n is denoted by $\mathcal{R}_n(k)$, and their intersection $\cap_k \mathcal{R}_n(k)$ is denoted by $\mathcal{R}_n(\infty)$. A matrix in the class $\mathcal{R}_n(\infty)$ is called *∞-rooted*. Clearly the following inclusions hold:

$$\mathcal{R}_n(\infty) \subseteq \cdots \subseteq \mathcal{R}_n(2) \subseteq \mathcal{R}_n(1) \subseteq \mathcal{CP}_n \subseteq \mathcal{DNN}_n.$$

Notice that, given a matrix $A \in \mathcal{DNN}_n$, it is immediate to check that $A \in \mathcal{CP}_n \setminus \mathcal{R}_n(1)$ implies that, for any fixed $k > 1$, $A^{2^k} \in \mathcal{R}_n(k-1) \setminus \mathcal{R}_n(k)$, hence a proper inclusion $\mathcal{R}_n(1) \subset \mathcal{CP}_n$ gives rise to an infinite strictly descending chain

$$\mathcal{R}_n(\infty) \subset \cdots \subset \mathcal{R}_n(2) \subset \mathcal{R}_n(1) \subset \mathcal{CP}_n.$$