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Extremal values of the trace norm over oriented trees

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ABSTRACT

The trace norm of the digraph D is defined as $\mathcal{N}(D) = \sum_{i=1}^{n} \sigma_i$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values of the adjacency matrix A of D, i.e. the square roots of the eigenvalues of $AA^\top$. We find the extremal values of $\mathcal{N}$ over the set of oriented trees and over the set of oriented paths.

1. Introduction

For concepts and definitions on digraphs we refer the reader to [2]. Our digraphs are simple, i.e. they have no loops nor multiple arcs. Assume that D is a digraph with set of vertices $\{1, \ldots, n\}$. The adjacency matrix $A = (a_{ij})$ of D is the $n \times n$ matrix defined as $a_{ij} = \begin{cases} 1 & \text{if } ij \text{ is an arc of } D \\ 0 & \text{if } ij \text{ is not an arc of } D \end{cases}$. The trace norm of D [7] is defined as

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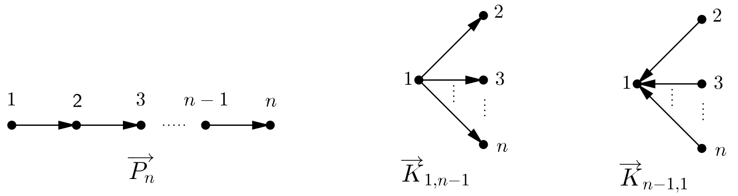

Fig. 1. Oriented trees where $\mathcal{N}$ attains extremal values.

$\mathcal{N}(D) = \sum_{i=1}^{n} \sigma_i$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values of A, i.e. the square roots of the eigenvalues of $AA^\top$. When D is a symmetric digraph and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A then $\sigma_i = |\lambda_i|$ for all $i = 1, \ldots, n$ and $\mathcal{N}(D) = \sum_{i=1}^{n} |\lambda_i|$, the energy introduced by Gutman [4]. For further details on the energy of graphs we refer to [6].

In a recent paper the study of mathematical properties of $\mathcal{N}$ over the set of digraphs was considered [1]. Specifically, lower and upper bounds for $\mathcal{N}$ over the set of digraphs were found and a comparison between $\mathcal{N}$ and the energy of digraphs $\mathcal{E}$ introduced in [8] was presented. In this note we study the trace norm over the set of oriented trees. Recall that an orientation of a graph G is a digraph obtained from G by replacing every edge uv of G by exactly one of the two arcs uv or vu. An oriented graph is an orientation of a graph. Similarly, an oriented tree is an orientation of a tree. We will show that among all oriented trees with n vertices, the minimal value of $\mathcal{N}$ is attained in exactly the two oriented star trees $\overrightarrow{K}_{1,n-1}$ and $\overrightarrow{K}_{n-1,1}$. The maximal value is attained in the oriented path $\overrightarrow{P_n}$ (see Fig. 1).

We also consider the extremal value problem of $\mathcal{N}$ over oriented paths. In fact we show that $\mathcal{N}$ reaches its minimal value in the oriented trees shown in Fig. 5.

2. Extremal values of the trace norm over oriented trees

We will denote by $\mathcal{OT}(n)$ the set of all oriented trees with n vertices. First we give some examples of oriented graphs and their trace norms which will be of importance in our study of extremal values of $\mathcal{N}$ over $\mathcal{OT}(n)$.

Example 2.1. Let $\overrightarrow{P_n}$ be the orientation of the path P_n shown in Fig. 1. If A is the adjacency matrix of $\overrightarrow{P_n}$ then clearly $AA^\top = \begin{pmatrix} \mathbf{I}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} \\ \mathbf{0}_{1\times(n-1)} & 0 \end{pmatrix}$. It follows that the singular values of $\overrightarrow{P_n}$ are $\sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} = 1$ and $\sigma_n = 0$. Hence $\mathcal{N}\left(\overrightarrow{P_n}\right) = n - 1$.

Example 2.2. Let K_{rs} be the complete bipartite graph with vertex partition $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_s\}$. Consider the (asymmetric) digraph $\overrightarrow{K}_{rs}$ obtained from K_{rs} by giving the following orientation: each edge $x_i y_j$ of K_{rs} is changed by an