Linear Algebra and its Applications 505 (2016) 269–281

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Near-invariant subspaces for matrix groups are nearly invariant

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ARTICLE INFO

Article history:
Received 22 August 2015
Accepted 3 May 2016
Available online 10 May 2016
Submitted by R. Brualdi

MSC:
15A30
20G20
47D99

Keywords:
Group
Semigroup
Reducibility
Invariant subspaces

ABSTRACT

Let $\mathcal{S}$ be a semigroup of invertible matrices. It is shown that if P is an idempotent matrix of rank and co-rank at least two such that the rank of $(1-P)SP$ is never more than one for S in $\mathcal{S}$ (the range of the kind of P is said to be near-invariant), then $\mathcal{S}$ has an invariant subspace within one dimension of the range of P (the kind of range is said to be nearly invariant).

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[1] Supported by a grant 371994-2014 from NSERC, Canada.
[2] Supported by a grant from the Slovenian Research Agency – ARRS.
[3] Supported by a grant from NSERC, Canada.

http://dx.doi.org/10.1016/j.laa.2016.05.005

1. Introduction

Let $\mathbb{F}$ be a field and consider a subspace $\mathcal{M}$ of $\mathbb{F}^n$. The invariance of $\mathcal{M}$ under a set $\mathcal{S}$ of linear operators on $\mathbb{F}^n$ is of course equivalent to the identity $(1-P)SP = 0$ for every S in $\mathcal{S}$, where P is an idempotent with range $\mathcal{M}$. We propose to study an approximate version of this for multiplicative semigroups of invertible operators – in particular, a group of operators – on $\mathbb{F}^n$:

Let $\mathcal{S}$ be such a semigroup. Let P be an idempotent matrix with range $\mathcal{M}$ in $\mathbb{F}^n$. Let $\mathcal{M}$ be a near-invariant subspace under $\mathcal{S}$, in the sense that the rank of $(1-P)SP$ is at most one for every S in $\mathcal{S}$. For this to be a meaningful condition (and to make the anticipated conclusion non-trivial) we impose the obvious restriction that both P and $I-P$ have rank at least two. We show that then $\mathcal{M}$ is nearly invariant in the following sense: there is a subspace $\mathcal{M}'$ of $\mathbb{F}^n$, invariant under every member of $\mathcal{S}$ such that either $\mathcal{M}'$ is a subspace of $\mathcal{M}$ with codimension at most one in $\mathcal{M}$ or the other way around. The methods are elementary, but not straight-forward; the arguments seem to be too tight for any further simplification.

We extend this result to include semigroups of quasi-affinities on an infinite-dimensional Hilbert space (i.e., injective bounded operators with dense range) so long as either $\mathcal{M}$ or its complement is of finite dimension. We also illustrate why the results are in some sense optimal.

It should be mentioned that if the collection of operators under consideration has more structure, better results can be obtained, as expected. If for example, $\mathcal{A}$ is an algebra of operators (as opposed to a group or a semigroup) on $\mathbb{C}^n$ (or any $\mathbb{F}^n$ with algebraically closed field $\mathbb{F}$) and P is a projection such that

$$\operatorname{rank}(I-P)\mathcal{A}P < \min\{\operatorname{rank}P, \operatorname{rank}(I-P)\},$$

then $\mathcal{A}$ has a nontrivial invariant subspace, because, otherwise $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ by Burnside's Theorem, and thus $(I-P)\mathcal{A}P$ contains many operators of the same rank as that of P or of $I-P$, whichever is smaller.

For infinite dimensions, also, stronger conclusions are known to be true. It is proved in [1], for example, that if $\mathcal{A}$ is an algebra of bounded operators on a Hilbert space $\mathcal{H}$ and if P is a projection on $\mathcal{H}$ with $\operatorname{rank}P = \operatorname{rank}(I-P) = \infty$, and if there is an integer m such that $\operatorname{rank}(I-P)AP \leqslant m$ for every $A \in \mathcal{A}$, then there is a (nontrivial) $\mathcal{A}$-invariant subspace $\mathcal{M}$ at most m dimensions apart from $P\mathcal{H}$. At the end of the present paper we give a modest extension of this result, Theorem 12.

We should point out here that even for a single operator A, the existence of such a projection P as above was proved only recently. The most definitive result is that of Popov and Tcaciuc [2]: For every A there is such a P with $\operatorname{rank}(I-P)AP \leqslant 1$. The well-known invariant subspace problem is whether for every A there is such a projection P, $0 \neq P \neq I$ with $(I-P)AP = 0$.