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The nowhere-zero eigenbasis problem for a graph

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ABSTRACT

Using the implicit function theorem it is shown that for any n distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, and for each connected graph G of order n, there is a real symmetric matrix A whose graph is G, the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, and every entry in each eigenvector of A is nonzero.

1. Introduction

The *graph of* an $n \times n$ real symmetric matrix $A = [a_{ij}]$ is the (simple) graph G on n vertices $1, 2, \ldots, n$ with edges $\{i, j\}$ if and only if $a_{ij} \neq 0$ and $i \neq j$. In the recent years considerable research has concerned the relationship between the spectrum of a symmetric matrix and its graph (for example see [3] and the references therein). The

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first result we recall asserts that every graph realizes each spectrum consisting of distinct eigenvalues. We denote the multi-set of eigenvalues of A by spec(A).

Theorem 1.1. *[3, Theorem 2.2.1] Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of n distinct real numbers and G be a graph on n vertices. Then there is a real symmetric matrix A whose graph is G and spec$(A) = \Lambda$.*

The nowhere-zero eigenbasis problem for G, raised by Shaun Fallat [2], is an extension of the Theorem 1.1 that puts extra requirements on the matrix A, namely that none of its eigenvectors has a zero entry. Note that if G is not connected, then A will be a direct sum of matrices and hence its eigenvectors will have zero entries. Thus, it is necessary to assume G is connected. More formally, the problem we study in this paper is the following.

> **The nowhere-zero eigenbasis problem for G.** For a given connected graph G on n vertices and given list $\lambda_1, \lambda_2, \ldots, \lambda_n$, of n distinct real numbers, does there exist a real symmetric matrix A whose graph is G, its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$, and none of the eigenvectors of A has a zero entry?

For a square matrix A and subsets α and β of indices, $A[\alpha, \beta]$ is the submatrix of A with rows indexed by α and columns indexed by β. The matrix obtained from A by deleting its j-th row and j-th column is denoted by $A(j)$. Note that if the j-th entry of an eigenvector of A is zero, then A and $A(j)$ share the eigenvalue corresponding to that eigenvector. The converse is also true; namely, if A and $A(j)$ share an eigenvalue λ, then there is an eigenvector of A corresponding to λ whose j-th entry is zero. To see this, assume $j = 1$ and note that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvectors of $A(1)$ and A, respectively, corresponding to the eigenvalue λ, then either the first entry of $\boldsymbol{x}$ is zero, or $A[\{2, \ldots, n\}, \{1\}]$ is in the column space of $A(1) - \lambda I$, which along with the symmetry of A imply that

$$\begin{bmatrix} 0 \\ \boldsymbol{y} \end{bmatrix}$$

is an eigenvector of A corresponding to λ. From this perspective, the nowhere-zero eigenbasis problem concerns the existence of a real symmetric matrix A with prescribed spectrum and graph such that $\sigma(A) \cap \sigma(A(j)) = \emptyset$ for each j.

The *Cauchy interlacing inequalities* guarantee that $\sigma(A(j))$ interlaces $\sigma(A)$, that is, if $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are eigenvalues of A and $\mu_1 \le \mu_2 \le \cdots \le \mu_{n-1}$ are eigenvalues of $A(j)$, then

$$\lambda_i \le \mu_i \le \lambda_{i+1}, \text{ for } i = 1, 2, \ldots, n-1. \tag{1}$$

Thus the condition $\sigma(A) \cap \sigma(A(j)) = \emptyset$ is equivalent to the condition that $\sigma(A(j))$ strictly interlaces $\sigma(A)$, that is, all the inequalities in (1) are strict. Hence, from this perspective,