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Real rank with respect to varieties

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ABSTRACT

We study the real rank of points with respect to a real variety X. This is a generalization of various tensor ranks, where X is in a specific family of real varieties like Veronese or Segre varieties. The maximal real rank can be bounded in terms of the codimension of X only. We show constructively that there exist varieties X for which this bound is tight. The same varieties provide examples where a previous bound of Blekherman–Teitler on the maximal X-rank is tight. We also give examples of varieties X for which the gap between maximal complex and the maximal real rank is arbitrarily large. To facilitate our constructions we prove a conjecture of Reznick on the maximal real symmetric rank of symmetric bivariate tensors. Finally we study the geometry of the set of points of maximal real rank in the case of real plane curves.

0. Introduction

The problem of decomposing a vector as a linear combination of simple vectors is central in many areas of applied mathematics, machine learning, and engineering. The length of the shortest decomposition is usually called the **rank** of the vector.

We consider the situation where the simple vectors form an **algebraic variety**. This includes well-studied and important cases such as tensor rank (real or complex), which

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is the rank with respect to the Segre variety, symmetric tensor rank (or Waring rank), which is the rank with respect to the Veronese variety, and anti-symmetric tensor rank, which is the rank with respect to the Grassmannian in its Plücker embedding.

We will be mostly interested in the rank over the real numbers. However, we make the definition over an arbitrary field $\mathbb{F}$. Let $X \subset \mathbb{P}^n$ be a projective variety defined over a field $\mathbb{F}$ and let $\hat{X} \subset \mathbb{A}^{n+1}$ be the affine cone over X. The variety X is called nondegenerate if X (or equivalently $\hat{X}$) is not contained in any hyperplane. In this case, for any $p \in \mathbb{F}^{n+1}$, $p \neq 0$, we can define the **rank** of p **with respect to** X (X-rank for short) as follows:

$$\operatorname{rank}_X(p) = \min r \quad \text{such that} \quad p = \sum_{i=1}^{r} x_i \quad \text{with} \quad x_1, \ldots, x_r \in \hat{X}(\mathbb{F}),$$

i.e. the rank of p with respect to X is the smallest length of an additive decomposition of p into points of $\hat{X}$ with coordinates in $\mathbb{F}$.

A rank r is called **generic** if the vectors of X-rank r contain a Zariski open subset of $\mathbb{A}^{n+1}$. Over any algebraically closed field, there is a unique generic X-rank for a nondegenerate variety X.

Over the real numbers, a rank r is called **typical** if the set of vectors of X-rank r contains an open subset of $\mathbb{R}^{n+1}$ with respect to the Euclidean topology. There can be many typical ranks for a given variety X.

In this paper, we study upper bounds for the maximal real rank and the range between the minimal typical rank and the maximal real rank in terms of the variety X which we allow to vary. We now outline our results:

Let r_0 and $r_{\max}$ denote the minimal typical rank and the maximal X-rank respectively. A trivial upper bound on the rank of a point is simply the dimension of the ambient vector space, which works over any field $\mathbb{F}$:

$$r_{\max} \leq n + 1. \tag{1}$$

In [13], Landsberg and Teitler showed that over the complex numbers (or any algebraically closed field) this bound can be improved:

$$r_{\max} \leq n - \dim X + 1 = \operatorname{codim} X + 1. \tag{2}$$

In [2], Blekherman and Teitler showed that for real varieties

$$r_{\max} \leq 2r_0, \tag{3}$$

and also

$$r_0 = r_{\text{gen}},$$

where r_{gen} is the generic rank with respect to X over $\mathbb{C}$.