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On a result of J.J. Sylvester

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ABSTRACT

For any algebraically closed field F and any two square matrices A, B over F, Sylvester (1884) [8] and Cecioni (1910) [1] showed that $AX = XB$ implies $X = 0$ if and only if A and B have no common eigenvalue. It is proved that a third equivalent statement is that, for any given polynomials f, g in $F[t]$, there exists h in $F[t]$ such that $f(A) = h(A)$ and $g(B) = h(B)$. Corresponding results hold also for any finite set of square matrices over F, and these lead to a new property of all associative rings and algebras (even over arbitrary fields) with 1.

1. Introduction

Our starting point here is the following result:

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Theorem 1.1. *For any field F, any $r, s \in \mathbb{N}$ and any square matrices $A \in M_r(F)$, $B \in M_s(F)$ whose eigenvalues all lie in F, the following three properties of the pair A, B are equivalent:*

- (i) *for any given polynomials $f, g \in F[t]$ there exists $h \in F[t]$ such that $f(A) = h(A)$ and $g(B) = h(B)$;*
- (ii) *A and B share no eigenvalue in common;*
- (iii) *for $r \times s$ matrices X over F, $AX = XB \Rightarrow X = 0$.*

The equivalence of (ii) and (iii) is an old result due to J.J. Sylvester [8] and F. Cecioni [1, p. 43] (or see [6, p. 90, Theorem 46.2]), while that of (i) and (ii) seems to be new. Since the arguments of Sylvester and Cecioni involve the unnecessary complications of resultants and canonical forms, while more modern proofs of (ii) ≡ (iii) use the Kronecker sum $I_s \otimes A - B^T \otimes I_r$ (see e.g. [9, p. 41, Theorem 2.7], and also well known to Sylvester), perhaps a simpler and more economical proof of the whole of Theorem 1.1 may now be worth putting on record. We give the details in Section 2.

In Section 3 we extend Theorem 1.1 more generally (Theorem 3.1) to finite sets $A_1, \ldots, A_m$ of square matrices (and discuss possible applications). To explain a slight discrepancy between the forms of (iii) in Theorems 1.1 and 3.1, we note here an obvious consequence of Theorem 1.1:

Corollary 1.2. *For any F, A, B as in Theorem 1.1,*

$$AX = XB \Rightarrow X = 0 \text{ if and only if } BY = YA \Rightarrow Y = 0. \qquad \square$$

In Section 4 we show that the case $r = s = n$ of Theorem 1.1 (and the case $n_1 = \ldots = n_m = n$ of Theorem 3.1) extends, in part, to associative algebras R more general than $M_n(F)$. Our main result (Theorem 4.1) is that in fact (i) ⇒ (iii) holds (even for infinite subsets of R) for *all* associative rings and algebras R with 1, but easy examples show that, for such R, (iii) no longer implies (i) and Corollary 1.2 also fails.

In view of the scarcity of known properties of arbitrary subsets valid for the class of all associative rings and algebras with 1, any new result of this generality might ordinarily be of considerable interest. However, since the proof of Theorem 4.1 is so transparent, and since neither of the properties (i) or (iii) seems at first sight to be of compelling interest in itself or to be connected to anything more familiar or more significant, Theorem 4.1 may be only a curiosity.

2. Proof of Theorem 1.1

(i) ⇒ (ii). If (ii) is false, then A, B have some common eigenvalue λ; let u, v be any corresponding pair of eigenvectors, and let f, g be the constant polynomials $f = 0$, $g = 1$. Then (i) would require $h(\lambda)u = h(A)u = f(A)u = 0$, so that $h(\lambda) = 0$, but also $h(\lambda)v = h(B)v = g(B)v = v$, so that $h(\lambda) = 1$, a contradiction. Hence (i) is false.