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Orientation preserving Möbius transformations in $\mathbb{R}^4_\infty$ and quaternionic determinants

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ABSTRACT

This paper explores $M(\mathbb{R}^4_\infty)$, the group of orientation preserving Möbius transformations acting in $\mathbb{R}^4_\infty$. On the one hand $M(\mathbb{R}^4_\infty)$ is given by the group of 2×2 matrices over the quaternions $\mathbb{H}$ with determinant $\mathcal{D}$ derived from the corresponding 4×4 matrices over the complex numbers $\mathbb{C}$. On the other hand we know that, in general, $M(\mathbb{R}^n_\infty)$ may be given in terms of 2×2 matrices over the Clifford algebra $\mathcal{C}_n$ with $n-1$ generators. Thus when $n=4$, $M(\mathbb{R}^4_\infty)$ is given in terms of 2×2 matrices with entries drawn from $\mathcal{C}_4$ and determinant Δ defined in terms of the entries of the given matrix. We note that the skew field $\mathbb{H}$ may be considered as a Clifford algebra $\mathcal{C}_3$ based on two generators i and j or more generally i_1 and i_2 where $ij = k$ or $i_1 i_2 = k$, while the set of elements $\{1, i, j, k\}$ form a basis of $\mathbb{H}$ regarded as a 4-dimensional real vector space. Thus $\mathbb{H}$ is embedded in $\mathcal{C}_4$. In the paper we reconcile the two representations of $M(\mathbb{R}^4_\infty)$ by comparing the generating sets of the underlying groups of matrices. A relationship between a determinants $\mathcal{D}$ and Δ is also exposed.

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1. Introduction

This paper explores $M(\mathbb{R}^4_\infty)$, the group of orientation preserving Möbius transformations acting in $\mathbb{R}^4_\infty$. Recall that the group of Möbius transformations acting in $\mathbb{R}^n_\infty$ is called the General Möbius group and is denoted by $GM(\mathbb{R}^n_\infty)$ [4]. Its elements are finite compositions of inversions (reflections) in hyperplanes and hyperspheres acting in $\mathbb{R}^n_\infty$. The Möbius group $M(\mathbb{R}^n_\infty)$ acting in $\mathbb{R}^n_\infty$ is a subgroup of $GM(\mathbb{R}^n_\infty)$ consisting of all orientation preserving Möbius transformations in $GM(\mathbb{R}^n_\infty)$. We know from [4] that the map $T : \mathbb{R}^n_\infty \to \mathbb{R}^n_\infty$ is in $GM(\mathbb{R}^n_\infty)$ if and only if it preserves cross-ratios $[x, y, u, v] = \dfrac{|x-u||y-v|}{|x-y||u-v|}$.

It is seen that when $n = 4$ there are two other ways of describing $M(\mathbb{R}^4_\infty)$. From Bisi [6], Kellerhals [8] and Wilker [21] we see that $M(\mathbb{R}^4_\infty)$ is given by the group of 2×2 matrices over the quaternions $\mathbb{H}$ with determinant $\mathcal{D}$ derived from the corresponding 4×4 matrices over the complex numbers $\mathbb{C}$. Wilker motivates this approach by noting that Möbius transformations acting in $\mathbb{R}^4_\infty$ have a Poincaré extension to Möbius transformations in $\mathbb{R}^5_\infty$ and these act as isometries of hyperbolic 5-space modeled in the upper half space. This is analogous to the way in which the Möbius transformations in $\mathbb{R}^2_\infty$ have a Poincaré extension to Möbius transformations in $\mathbb{R}^3_\infty$ and act as isometries in hyperbolic 3-space modeled in the upper half space $\{(x, y, t) : t > 0\}$. He also notes that since the points of $\mathbb{R}^4$ can be represented by arbitrary quaternions, any proper orthogonal transformation can be expressed using a suitable pair of unit quaternions u and v by the mapping $q \mapsto uq\overline{v}$. This representation is analogous to the representation of rotations through 2ψ in $\mathbb{R}^3$ about an axis with direction cosines given by I and with the formula $q \mapsto wq\overline{w}$ where w is a unit quaternion and $w = cos\psi + sin\psi\mathrm{I}$ [7].

On the other hand there is a more general approach to this group of Möbius maps. In Ahlfors [1] and Waterman [20] we find this general approach in terms of Clifford matrices. Ahlfors notes that this method was introduced by K.T. Vahlen as early as 1901 [18]. While the method is also modeled on the complex case, its approach is entirely different. The aim of this approach is to unify the theory of motions in Euclidean, hyperbolic and elliptic space and uses Clifford numbers for the study of Möbius transformations. In general, $M(\mathbb{R}^n_\infty)$ may be given in terms of 2×2 matrices over the Clifford algebra $\mathcal{C}_n$ with $n-1$ generators. Thus when $n = 4$, $M(\mathbb{R}^4_\infty)$ is given in terms of 2×2 matrices with entries drawn from $\mathcal{C}_4$ with a determinant Δ defined in terms of these entries.

In the existing literature the isomorphism between the two groups representing $M(\mathbb{R}^4_\infty)$ is not made explicit. It is the intention of this paper to specify this isomorphism and thus to clarify this situation. Through this process the relationship between the quaternionic determinants Δ and $\mathcal{D}$ is also exposed. We note that the skew field $\mathbb{H}$ may be considered as a Clifford algebra $\mathcal{C}_3$ based on two generators i and j or more generally i_1 and i_2 where $ij = k$ or $i_1 i_2 = k$, while the set of elements $\{1, i, j, k\}$ form a basis of $\mathbb{H}$ regarded as a 4-dimensional real vector space. Thus $\mathbb{H}$ is embedded in $\mathcal{C}_4$ as a Clifford algebra and is isomorphic to $\mathbb{R}^4$ as a real vector space. It is further noted that