Linear Algebra and its Applications 507 (2016) 77–95

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Row sums and alternating sums of Riordan arrays

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A R T I C L E   I N F O

Article history:
Received 28 October 2015
Accepted 28 May 2016
Available online 3 June 2016
Submitted by R. Brualdi

MSC:
05A15
05A05
11B39
11B73
15B36
15A06
05A19
11B83

Keywords:
Riordan arrays
Riordan group
Appell subgroup
Associate subgroup
Bell subgroup
Derivative subgroup
Checkerboard subgroup
Hitting-time subgroup
Girard–Waring type identities
Row sum functions and alternating row sum functions of Riordan arrays

A B S T R A C T

Here we use row sum generating functions and alternating sum generating functions to characterize Riordan arrays and subgroups of the Riordan group. Numerous applications and examples are presented which include the construction of Girard–Waring type identities. We also show the extensions to weighted sum (generating) functions, called the expected value (generating) functions of Riordan arrays.

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http://dx.doi.org/10.1016/j.laa.2016.05.035

1. Introduction

Riordan arrays are infinite, lower triangular matrices defined by two generating functions. They form a group, called the Riordan group (see Shapiro et al. [15]). More formally, consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the order of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$. The set of formal power series of order r is denoted by $\mathcal{F}_r$. It is known that $\mathcal{F}_0$ is the set of invertible f.p.s. and $\mathcal{F}_1$ is the set of compositionally invertible f.p.s., that is, the f.p.s. $f(t)$ for which the compositional inverse $\bar{f}(t)$ exists such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the (proper) Riordan array $D = (d_{n,k})_{n,k\geq 0} = (d(t), h(t))$, where

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, $d(t)h(t)^k$ is the generating function for the entries of column k.

Let $[f_0, f_1, f_2, \ldots]^T$ be a column vector with $f(t) = \sum_{n\geq 0} f_n t^n$. It is convenient to switch freely between a sequence, a sequence written as a column vector, and the ordinary generating function for that sequence. We then have the fundamental theorem of Riordan arrays

$$(d(t), h(t))\,[f_0, f_1, f_2, \ldots]^T = (d(t),\, h(t))f(t) = d(t)f(h(t)). \tag{2}$$

It follows quickly that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\, h_1(t)) * (d_2(t),\, h_2(t)) = (d_1(t)d_2(h_1(t)),\, h_2(h_1(t))). \tag{3}$$

The Riordan array $I = (1,\, t)$ is everywhere 0 except for all 1's on the main diagonal; it can be easily proved that I acts as an identity for this product, that is, $(1,\, t) * (d(t),\, h(t)) = (d(t),\, h(t)) * (1, t) = (d(t),\, h(t))$. Let $(d(t), h(t))$ be a Riordan array. Then its inverse is

$$(d(t),\, h(t))^{-1} = \left(\frac{1}{d(\bar{h}(t))},\, \bar{h}(t)\right), \tag{4}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays forms a group (see [15]).

Multiplying a matrix by the column vector $[1, 1, 1, \ldots]^T$ yields the column of row sums which we denote as R^+. Since $f(t) = 1/(1-t)$ is the corresponding generating function, then (2) presents the generating function R^+ of the row sum sequence of the Riordan array $(d(t),\, h(t))$ (see [14]), i.e.,

$$R^+(t) := (d(t), h(t))\frac{1}{1-t} = \frac{d(t)}{1-h(t)}. \tag{5}$$