

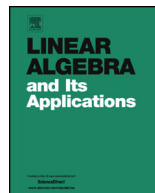


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## Commuting maps over the ring of strictly upper triangular matrices



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### ABSTRACT

Let  $N_r$ ,  $r \geq 4$ , be the ring of strictly upper triangular matrices with entries in a field  $F$  of characteristic zero. We describe all linear maps  $f : N_r \rightarrow N_r$  satisfying  $[f(x), x] = 0$  for every  $x \in N_r$ .

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For a ring  $R$  we say that the map  $f : R \rightarrow R$  is *commuting* if  $[f(x), x] = 0$  for every  $x \in R$  where  $[a, b] = ab - ba$  denotes the standard commutator. The first general result regarding commuting maps comes from Brešar [2] when it was shown that additive commuting maps  $f$  over a simple unital ring  $R$  must be of the form  $f(x) = \lambda x + \mu(x)$  for some  $\lambda \in Z(R)$  and additive  $\mu : R \rightarrow Z(R)$  where  $Z(R)$  denotes the center of  $R$ . This form is usually called a *standard form* for the commuting map. There are plenty of results on commuting maps and the reader is referred to the survey paper [3] for acquaintance with the development of the theory of commuting maps and the various results that have been established.

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In 2000, Beidar, Brešar, and Chebotar [1] proved that a similar result holds true over  $T_r = T_r(F)$ , the ring of  $r \times r$  upper triangular matrices over a field  $F$ . Their work showed that any linear commuting map  $f : T_r \rightarrow T_r$  is again of the standard form, so  $f(x) = \lambda x + \mu(x)$  for some  $\lambda \in F$  and linear map  $\mu : T_r \rightarrow Z(T_r)$ .

In this short note, we examine commuting maps over the ring of strictly upper triangular matrices  $N_r = N_r(F)$  over a field  $F$ . It is well-known that the center  $Z(N_r)$  consists of matrices of the form  $\{ze_{1,r}\}$ , where  $z \in F$  and  $e_{1,r}$  is a matrix unit. It would be natural to expect that the commuting maps on  $N_r$  are of the standard form. However, to our surprise it is not the case. Fortunately, their description still looks nice as commuting maps are “almost” of the standard form.

**Theorem 1.** *Let  $r \geq 4$  and let  $N_r = N_r(F)$  be the ring of strictly upper triangular matrices over a field  $F$  of characteristic zero. Let  $f : N_r \rightarrow N_r$  be a commuting linear map. Then there exist  $\lambda \in F$  and an additive map  $\mu : N_r \rightarrow \Omega$  such that  $f(x) = \lambda x + \mu(x)$  for all  $x \in N_r$  where  $\Omega = \{ae_{1,r-1} + be_{1,r} + ce_{2,r} : a, b, c \in F\}$  and  $e_{i,j}$  denotes the standard matrix unit.*

The set  $\Omega$  in the statement of Theorem 1 is essential for our conclusion. Our next example demonstrates that it is easy to find a commuting map over  $N_r$  that is not of the standard form.

**Example 2.** Consider the map  $G : N_r \rightarrow N_r$  defined as follows:

$$G : \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,r} \\ & 0 & a_{2,3} & \cdots & a_{2,r} \\ & & 0 & \ddots & \vdots \\ & & & 0 & a_{r-1,r} \\ & & & & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \cdots & 0 & a_{1,2} & 0 \\ & 0 & \cdots & 0 & a_{r-1,r} \\ & & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}.$$

Then for  $A = (a_{i,j}) \in N_r$  we have that  $G(A)A = a_{1,2}a_{r-1,r}e_{1,r} = AG(A)$ , making  $G$  a commuting map on  $N_r$  of the form outlined in Theorem 1 with  $\lambda = 0$ .

Throughout the rest of the paper we will assume that the field  $F$  is of characteristic zero. In order to prove our main result, we first need to establish a few lemmas. Given that we are examining commuting maps, it is considerably helpful to be able to describe the centralizer of each element in  $N_r$ . We then make use of this first lemma, which follows from Theorem 3.2.4.2 of Horn and Johnson [4].

**Lemma 3.** *Let  $A = \sum_{i=1}^{r-1} a_{i,i+1}e_{i,i+1}$  such that  $a_{i,i+1} \neq 0$  for  $1 \leq i \leq r-1$ . Then the centralizer of  $A$  in  $N_r$  is given by  $C_A = \{\alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{r-1} A^{r-1}\}$ .*

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