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Commuting maps over the ring of strictly upper triangular matrices

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A B S T R A C T

Let N_r, $r \geq 4$, be the ring of strictly upper triangular matrices with entries in a field F of characteristic zero. We describe all linear maps $f : N_r \to N_r$ satisfying $[f(x), x] = 0$ for every $x \in N_r$.

For a ring R we say that the map $f : R \to R$ is *commuting* if $[f(x), x] = 0$ for every $x \in R$ where $[a, b] = ab - ba$ denotes the standard commutator. The first general result regarding commuting maps comes from Brešar [2] when it was shown that additive commuting maps f over a simple unital ring R must be of the form $f(x) = \lambda x + \mu(x)$ for some $\lambda \in Z(R)$ and additive $\mu : R \to Z(R)$ where $Z(R)$ denotes the center of R. This form is usually called a *standard form* for the commuting map. There are plenty of results on commuting maps and the reader is referred to the survey paper [3] for acquaintance with the development of the theory of commuting maps and the various results that have been established.

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In 2000, Beidar, Brešar, and Chebotar [1] proved that a similar result holds true over $T_r = T_r(F)$, the ring of $r \times r$ upper triangular matrices over a field F. Their work showed that any linear commuting map $f : T_r \to T_r$ is again of the standard form, so $f(x) = \lambda x + \mu(x)$ for some $\lambda \in F$ and linear map $\mu : T_r \to Z(T_r)$.

In this short note, we examine commuting maps over the ring of strictly upper triangular matrices $N_r = N_r(F)$ over a field F. It is well-known that the center $Z(N_r)$ consists of matrices of the form $\{ze_{1,r}\}$, where $z \in F$ and $e_{1,r}$ is a matrix unit. It would be natural to expect that the commuting maps on N_r are of the standard form. However, to our surprise it is not the case. Fortunately, their description still looks nice as commuting maps are "almost" of the standard form.

Theorem 1. *Let $r \geq 4$ and let $N_r = N_r(F)$ be the ring of strictly upper triangular matrices over a field F of characteristic zero. Let $f : N_r \to N_r$ be a commuting linear map. Then there exist $\lambda \in F$ and an additive map $\mu : N_r \to \Omega$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in N_r$ where $\Omega = \{ae_{1,r-1} + be_{1,r} + ce_{2,r} : a, b, c \in F\}$ and $e_{i,j}$ denotes the standard matrix unit.*

The set Ω in the statement of Theorem 1 is essential for our conclusion. Our next example demonstrates that it is easy to find a commuting map over N_r that is not of the standard form.

Example 2. Consider the map $G : N_r \to N_r$ defined as follows:

$$G : \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,r} \\ & 0 & a_{2,3} & \cdots & a_{2,r} \\ & & 0 & \ddots & \vdots \\ & & & 0 & a_{r-1,r} \\ & & & & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \cdots & 0 & a_{1,2} & 0 \\ & 0 & \cdots & 0 & a_{r-1,r} \\ & & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}.$$

Then for $A = (a_{i,j}) \in N_r$ we have that $G(A)A = a_{1,2}a_{r-1,r}e_{1,r} = AG(A)$, making G a commuting map on N_r of the form outlined in Theorem 1 with $\lambda = 0$.

Throughout the rest of the paper we will assume that the field F is of characteristic zero. In order to prove our main result, we first need to establish a few lemmas. Given that we are examining commuting maps, it is considerably helpful to be able to describe the centralizer of each element in N_r. We then make use of this first lemma, which follows from Theorem 3.2.4.2 of Horn and Johnson [4].

Lemma 3. *Let $A = \sum_{i=1}^{r-1} a_{i,i+1}e_{i,i+1}$ such that $a_{i,i+1} \neq 0$ for $1 \leq i \leq r-1$. Then the centralizer of A in N_r is given by $C_A = \{\alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{r-1}A^{r-1}\}$.*