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Avoiding singular coarse grid systems

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Here we consider the iterative solution of linear systems of equations with a symmetric positive semidefinite system matrix. If multilevel methods in combination with Krylov subspace methods are used to find the solution of these systems, often singular subsystems or coarse grid systems have to be solved. Then the Moore–Penrose inverse of the coarse grid systems can be used. Here, we establish some theoretical techniques how to avoid the singularity of the coarse grid system, while the resulting operator remains the same as we would have used the Moore–Penrose inverse. One option is to delete specific columns of the restriction and prolongation operator. The other option is to perturb the system matrix.

1. Introduction

Multigrid and multilevel methods in combination with Krylov subspace methods are the methods of choice to solve a linear system of the form

$$Ax = b, \quad A \in \mathbb{R}^{n\times n}, \tag{1.1}$$

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whose coefficient matrix A is sparse. The multilevel methods often act as a preconditioner for the Krylov subspace method. Stimulated by the numerical solution of PDEs and the problem of solving the resulting linear system, different multilevel or two-level preconditioners have been developed successfully during the last decades. These preconditioners use the solution of smaller systems, so-called Galerkin systems or coarse-grid systems. Well-known examples of these preconditioners are the geometric and algebraic multigrid methods (see, e.g., [22,23]) and domain decomposition methods with coarse-grid corrections, such as the BPS method [2,4] and the balancing Neumann–Neumann method [8–10,17,21]. In the same period, the augmented and deflated Krylov subspace methods were established [3,11,15,16], which can also be seen as two-level methods.

In [12–14,20], a detailed abstract comparison of some of these multilevel or coarse-grid correction and deflation methods applied to a symmetric positive definite (SPD) coefficient matrix A is presented. From an abstract point of view, there are many common ingredients. It has been shown that most of the above mentioned methods are strongly related to each other. Some of these methods are even mathematically equivalent. Each of these methods uses a projection of the form

$$P := I - AQ, \quad Q := ZE^{-1}Z^T, \quad E := Z^T AZ. \tag{1.2}$$

Here the matrix $Z \in \mathbb{R}^{n\times r}$ with $r \leq n$ describes an operator

$$Z : \mathbb{R}^r \mapsto \mathbb{R}^n.$$

In the multilevel language the operator Z is called the *prolongation* or *interpolation* operator, which transfers some unknowns at a coarse level to a fine one. We always assume that Z has full column rank. The *restriction* operator

$$Z^T : \mathbb{R}^n \mapsto \mathbb{R}^r$$

does the opposite; it transfers some unknowns at a fine level to a coarse one. The so called *Galerkin matrix* or *coarse grid matrix* is then defined by

$$E = Z^T AZ, \quad E \in \mathbb{R}^{r\times r}.$$

The projection P in (1.2) is then often used as a coarse grid correction operator.

In the deflation methods, the matrix Z consists of deflation vectors and Z is called the deflation subspace matrix. The matrix P is used as a projection which deflates some eigenvalues out of the spectrum of A (see Section 2).

But, whenever P is used in the above mentioned methods, a linear system – the coarse grid system – with coefficient matrix E has to be solved. In many applications the matrices A and E are nonsingular. However, linear systems with a singular symmetric positive semidefinite (SPSD) matrix appear frequently in practice, especially, in the numerical solutions of PDEs, if Neumann boundary conditions are involved. In these