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Counterexamples to the conjecture on stationary probability vectors of the second-order Markov chains

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ABSTRACT

It was conjectured in the paper "Stationary probability vectors of higher-order Markov chains" (Li and Zhang, 2015 [7]) that if the set of stationary vectors of the second-order Markov chain contains k-interior points of the $(k-1)$-dimensional face of the simplex Ω_n then every vector in the $(k-1)$-dimensional face is a stationary vector. In this paper, we provide counterexamples to this conjecture.

1. The m-order Markov chains

A discrete-time m-order Markov chain is a stochastic process with a sequence of random variables

$$\{X_t,\ t = 0, 1, 2 \ldots\},$$

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which takes on values in a discrete finite state space

$$[n] = \{1, \ldots, n\}$$

for a positive integer n, such that

$$\begin{aligned} p_{i,i_1\cdots i_m} &= \Pr\left(X_{t+1} = i | X_t = i_1, X_{t-1} = i_2, \ldots, X_1 = i_t, X_0 = i_{t+1}\right) \\ &= \Pr\left(X_{t+1} = i | X_t = i_1, \cdots, X_{t-m+1} = i_m\right), \end{aligned}$$

where $i, i_1, \cdots, i_m, \cdots, i_{t+1} \in [n]$ and

$$p_{i,i_1\cdots i_m} \geq 0, \qquad \sum_{i=1}^{n} p_{i,i_1\cdots i_m} = 1, \quad 1 \leq i, i_1, \cdots, i_m \leq n.$$

Namely, the current state of the chain depends on m past states (see [1,2]). Here $\mathcal{P} = (p_{i,i_1\cdots i_m})_{i,i_1,\ldots,i_m=1}^{n}$ is called *the transition hypermatrix* of an m-order Markov chain $\{X_t,\ t = 0, 1, 2\ldots\}$. Let $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))^T$ be the tth-distribution of the discrete-time m-order Markov chain $\{X_t,\ t = 0, 1, 2\ldots\}$. Note that $\mathcal{P}$ is *an* $(m+1)$*-order stochastic hypermatrix* governing the transition of states in the m-order Markov chain according to the following rule

$$x_i(t+1) = \sum_{1 \leq i_1, \cdots, i_m \leq n} p_{i,i_1\cdots i_m} x_{i_1}(t) \cdots x_{i_m}(t), \quad i = 1, \cdots, n.$$

Denote by

$$\Omega_n = \left\{ \mathbf{x} = (x_1, \cdots, x_n)^T : x_1, \cdots, x_n \geq 0,\ \sum_{i=1}^{n} x_i = 1 \right\} \tag{1}$$

the standard simplex consisting of probability vectors in $\mathbb{R}^n$.

A polynomial operator $\mathfrak{P} : \Omega_n \to \Omega_n$ associated with the $(m+1)$-order stochastic hypermatrix $\mathcal{P} = (p_{i,i_1\cdots i_m})_{i,i_1,\ldots,i_m=1}^{n}$

$$(\mathfrak{P}(\mathbf{x}))_i = \sum_{1 \leq i_1, \cdots, i_m \leq n} p_{i,i_1\cdots i_m} x_{i_1} \cdots x_{i_m}, \quad i = 1, \cdots, n, \tag{2}$$

is called *a nonlinear Markov operator* (see [3]). It is clear that a set of all stationary distributions of the m-order Markov chain is nothing but a set of all fixed points of the nonlinear Markov operator (2). Due to Brouwer's fixed-point theorem, the set $\mathbf{Fix}(\mathfrak{P}) = \{\mathbf{x} \in \Omega_n : \mathfrak{P}(\mathbf{x}) = \mathbf{x}\}$ is nonempty.

If $m = 2$ then $\mathfrak{P} : \Omega_n \to \Omega_n$ is *a quadratic stochastic operator* which has an incredible application in population genetics (see [4–6]).