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Line graphs and the transplantation method

Peter Herbrich

Department of Mathematics, Dartmouth College, Hanover, NH, USA

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We study isospectrality for mixed Dirichlet–Neumann boundary conditions and extend the previously derived graph-theoretic formulation of the transplantation method. Led by the theory of Brownian motion, we introduce vertex-colored and edge-colored line graphs that give rise to block diagonal transplantation matrices. In particular, we rephrase the transplantation method in terms of representations of free semigroups and provide a method for generating adjacency cospectral weighted directed graphs.

1. Introduction

Inverse spectral geometry studies the extend to which a geometric object, e.g., a Euclidean domain, is determined by the spectral data of an associated operator, e.g., the eigenvalues of the Laplace operator with suitable boundary conditions. This objective is beautifully summarized by Kac's influential question "Can one hear the shape of a drum?" [1]. Recently, the author [2] studied broken drums each of which is modeled as a compact flat manifold M with boundary $\partial M = \overline{\partial_D M \cup \partial_N M}$, where $\partial_D M$ and $\partial_N M$

E-mail address: peter.herbrich@dartmouth.edu.

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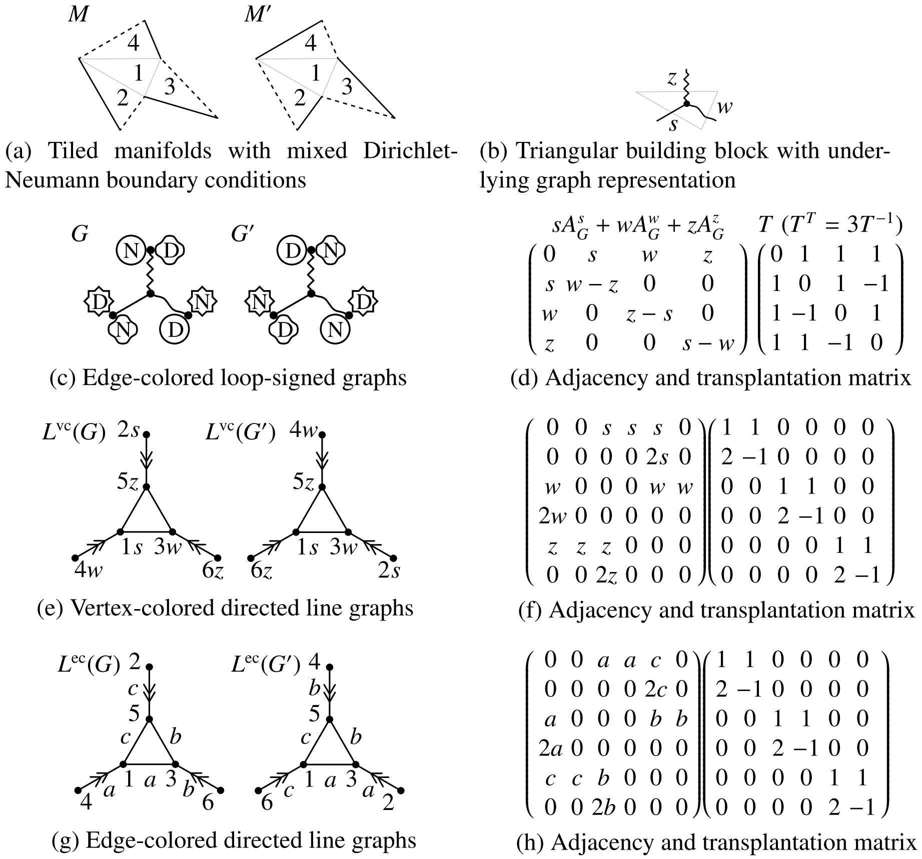

(a) Tiled manifolds with mixed Dirichlet-Neumann boundary conditions
(b) Triangular building block with underlying graph representation
(c) Edge-colored loop-signed graphs
(d) Adjacency and transplantation matrix
(e) Vertex-colored directed line graphs
(f) Adjacency and transplantation matrix
(g) Edge-colored directed line graphs
(h) Adjacency and transplantation matrix

Fig. 1. Graph representations of a pair of transplantable tiled manifolds with mixed Dirichlet–Neumann boundary conditions. Solid boundary segments carry Dirichlet conditions and dashed ones carry Neumann conditions. The adjacency matrices on the right belong to the respective first graph on the left. The types of line (*straight*, *wavy*, *zigzag*) represent the edge colors (s, w, z) of G and G'. The graphs $L^{\mathrm{ec}}(G)$ and $L^{\mathrm{ec}}(G')$ have edge colors $(a, b, c) = (\{s, w\}, \{w, z\}, \{s, z\})$.

represent the attached and unattached parts of the drumhead, respectively. The audible frequencies of such a broken drum are determined by the eigenvalues of the Laplace–Beltrami operator Δ_M of M with Dirichlet and Neumann boundary conditions along $\partial_D M$ and $\partial_N M$, respectively. Provided that ∂M is sufficiently smooth, this operator has discrete spectrum given by an unbounded non-decreasing sequence of non-negative eigenvalues.

Using number-theoretic ideas, Sunada [3] developed a celebrated method involving group actions to construct isospectral manifolds, i.e., manifolds whose spectra coincide. It ultimately allowed Gordon et al. [4] to answer Kac's question in the negative. Buser [5] distilled the combinatorial core of Sunada's method into the transplantation method, which involves tiled manifolds that are composed of identical building blocks, e.g., M and M' in Fig. 1a. In essence, if φ is an eigenfunction of Δ_M that satisfies the desired boundary conditions, then its restrictions $(\varphi_i)_{i=1}^{4}$ to the building blocks of M