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Sums of square-zero endomorphisms of a free module

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ABSTRACT

Let R be an associative ring with unity and let M be a free right R-module of infinite rank. We prove that any endomorphism of M can be written as a sum of four square-zero endomorphisms. This result is optimal.

1. Introduction

The problem of decomposing an endomorphism into a sum of square-zero endomorphisms is being studied for fifty years [2,4–6,8,10], and there exist several well-known results on this topic. The most widely studied case is that of endomorphisms of a finite-dimensional vector space, which correspond to $n \times n$ matrices over a field. An obvious

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necessary condition for a matrix A to be a sum of square-zero matrices is $\operatorname{tr}(A) = 0$, and this condition turns out to be sufficient as well. More than that, it is known that every matrix with zero trace is a sum of four square-zero matrices [7,10]. Also, the problem of describing the set of sums of two (or three) square-zero matrices has attracted a significant amount of attention [1,3,9]. Another direction of research is the study of bounded operators over Hilbert spaces [2,5,10,11]. It is known that every such operator A is a sum of five square-zero bounded operators; also, A can be written as a sum of four square-zero bounded operators if and only if A is a commutator [10]. Slowik [8] and de Seguins Pazzis [6] considered this problem in a different setting of column-finite matrices, that is, for endomorphisms of an infinite-dimensional vector space. This paper is devoted to a further generalization of this problem. Namely, we study the sums of square-zero endomorphisms of a free module over a ring. The question we are interested in can be formulated as follows.

Question 1. What is the smallest integer k such that, for every ring R and every right free R-module M of infinite rank, every endomorphism of M can be written as a sum of k square-zero endomorphisms of M?

Remark 2. Here and in what follows, the word *ring* means an associative ring with unity. In particular, in contrast with the above mentioned papers, we allow our structure to contain zero divisors.

A recent result by Slowik [8] states that every operator on a vector space of countably infinite dimension is a sum of ten square-zero operators. In other words, one has $k \leq 10$ in Question 1 if R is a field. The result of Slowik has been improved by de Seguins Pazzis [6], who showed that every operator on a vector space of infinite dimension is a sum of four square-zero operators. This result is optimal in the class of fields, that is, $k = 4$ is the answer to Question 1 if R is required to be a field. In our paper, we generalize these results and get a complete answer to Question 1.

Theorem 3. *Every endomorphism of a right free R-module of infinite rank is a sum of $k = 4$ square-zero endomorphisms.*

As we already mentioned, Theorem 3 would not be true with $k = 3$. Therefore, $k = 4$ is the answer to Question 1. Our goal is to prove Theorem 3.

2. Setting up

In this paper, R always denotes a ring and M a right free R-module of infinite rank. Let $V = (v_i)$ be a basis of M. Any endomorphism $f \in \operatorname{End}_R M$ can be specified by the images of the v_i's, that is, by the coefficients $F_{ji} \in R$ such that $f(v_i) = \sum_j v_j F_{ji}$. We get an infinite matrix F whose rows and columns are indexed with the elements