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Normal matrices subordinate to a tree and flat portions of the field of values

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A B S T R A C T

Matrices subordinate to trees are considered. An efficient normality characterization for any such matrix is given, and several consequences (not valid for general normal matrices) of it are established. In addition, the existence (and enumeration) of flat portions on the boundary of the field of values of matrices subordinate to a tree is characterized.

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1. Introduction

Let M_n denote the set of all n-by-n matrices with the entries in $\mathbb{C}$. A matrix $A = (a_{ij})_{i,j=1}^n \in M_n$ is *normal* if it commutes with its conjugate transpose A^*:

$$AA^* = A^*A. \tag{1.1}$$

The normality of A can be restated in many equivalent ways (see e.g. [6]), the most direct of which being the entry-wise form of (1.1):

$$\textstyle\sum_{k=1}^n a_{ik}\overline{a_{jk}} = \sum_{k=1}^n a_{kj}\overline{a_{ki}} \tag{1.2}$$

for $i, j = 1, \dots, n$.

It was observed in [2, Lemma 5.1] that for *tridiagonal* matrices

$$\begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & 0 & c_{n-1} & a_n \end{bmatrix} \tag{1.3}$$

the normality criterion (1.2) simplifies substantially. Namely:

Proposition 1. *The matrix (1.3) is normal if and only if $|b_j| = |c_j|$ for all $j = 1, \dots, n-1$, $\arg b_j + \arg c_j$ does not depend on j for all consecutive j such that $b_j \neq 0$, and $2\arg(a_{j+1} - a_j) = \arg b_j + \arg c_j$ whenever $a_j \neq a_{j+1}$.*

In particular, if the normal matrix (1.3) is unreduced, that is, there are no zero pairs $\{b_j, c_j\}$ among its off diagonal entries, then the value of $\arg b_j + \arg c_j$ is constant throughout; see [1, Lemma 1] for the precise statement and its applications.

A similar, but nevertheless somewhat different, result holds for *arrowhead matrices*

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & b_1 \\ 0 & a_2 & 0 & \cdots & b_2 \\ 0 & 0 & a_3 & \cdots & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & a_n \end{bmatrix}, \tag{1.4}$$

all non-zero entries of which are located on the main diagonal and in the last row and column. This fact was obtained in a Capstone project of the third author (supervised by the second author), and it is as follows: