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On the spectral characterization of pineapple graphs

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ABSTRACT

The pineapple graph K_p^q is obtained by appending q pendant edges to a vertex of a complete graph K_p ($q \geq 1$, $p \geq 3$). Zhang and Zhang (2009) [7] claim that the pineapple graphs are determined by their adjacency spectrum. We show that their claim is false by constructing graphs which are cospectral and non-isomorphic with K_p^q for every $p \geq 4$ and various values of q. In addition we prove that the claim is true if $q = 2$, and refer to the literature for $q = 1$, $p = 3$, and $(p, q) = (4, 3)$.

1. Introduction

The pineapple graph K_p^q is the coalescence of the complete graph K_p (at any vertex) with the star $K_{1,q}$ at the vertex of degree q. Thus K_p^q can be obtained from K_p by appending q pendant edges to a vertex of K_p. Clearly K_p^q has $n = p + q$ vertices, $\binom{p}{2} + q$ edges and $\binom{p}{3}$ triangles. In order to exclude the complete graphs and the stars, we assume $p \geq 3$ and $q \geq 1$. See Fig. 1 for a drawing of K_4^4. Alternatively, K_p^q can be defined by its adjacency matrix

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$$A = \begin{bmatrix} 0 & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{1} & J_{p-1} - I & O \\ \mathbf{1} & O & O \end{bmatrix},$$

where $\mathbf{1}$ is the all-ones vector (of appropriate size), and J_ℓ denotes the $\ell \times \ell$ all-ones matrix.

Proposition 1.1. *The characteristic polynomial* $p(x) = \det(xI - A)$ *of the pineapple graph* K_p^q *equals*

$$p(x) = x^{q-1}(x+1)^{p-2}(x^3 - x^2(p-2) - x(p+q-1) + q(p-2)).$$

Proof. The adjacency matrix A has q identical rows, so $\operatorname{rank}(A)$ is at most $p+1$, and therefore $p(x)$ has a factor x^{q-1}. Similarly, $A+I$ has $p-1$ identical rows and so $(x+1)^{p-2}$ is another factor of $p(x)$. The given partition of A is equitable with quotient matrix

$$Q = \begin{bmatrix} 0 & p-1 & q \\ 1 & p-2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(this means that each block of A has constant row sums, which are equal to the corresponding entry of Q). The characteristic polynomial of Q equals $q(x) = \det(xI - Q) = x^3 - x^2(p-2) - x(p+q-1) + q(p-2)$, and it is well known (see for example [3], or [1]) that $q(x)$ is a divisor of $p(x)$. $\square$

In this paper we deal with the question whether K_p^q is the only graph with characteristic polynomial $p(x)$. In other words, is K_p^q determined by its spectrum? Note that the complete graph K_n is determined by its spectrum, but for the star $K_{1,n-1}$ this is only the case when $n = 2$, or $n-1$ is a prime. In [7] it is stated that every pineapple graph is determined by its spectrum. The presented proof, however, is incorrect. Even worse, the result is false. In the next section we shall construct graphs with the same spectrum as K_p^q for every $p \geq 4$ and several values of q.

When $q = 1$, the pineapple graph K_p^1 can be obtained from the complete graph K_{p+1} by deleting the edges of the complete bipartite graph $K_{1,p-1}$. Graphs constructed in this way are known to be determined by their spectra, see [2].

Zhang and Zhang [7] proved (correctly this time) that the graph obtained by adding q pendant edges to a vertex of an odd circuit is determined by the spectrum of the adjacency matrix. When the odd circuit is a triangle we obtain that K_3^q is determined by its spectrum.

Godsil and McKay [6] generated by computer all pairs of non-isomorphic cospectral graphs with seven vertices. Since K_4^3 is not in their list, it is determined by its spectrum.

In Section 3 we prove that the spectrum determines K_p^q when $q = 2$. The proof uses the classification of graphs with least eigenvalue greater than -2.