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Characterization of solutions of non-symmetric algebraic Riccati equations

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ABSTRACT

We give a new characterization of the solution set of non-symmetric algebraic Riccati equations involving real matrices. Our characterization involves the use of invariant subspaces of the coefficient matrices. We also give a poset structure on the solutions of ARE and explore some properties of this poset (partially ordered set).

1. Introduction

We give a rank characterization of all the real solutions of non-symmetric algebraic Riccati equations (ARE) with real matrices. The main coefficients A, B of the considered AREs (Equation (1)) are square. We look at the ARE of the form

$$AK + KB + C - KDK = 0 \tag{1}$$

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where both K and $C \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{n\times m}$. Such AREs are known in the literature as non-symmetric AREs. Algebraic Riccati equations with symmetric solutions having square matrices (with $A = B^T$ and C, D being symmetric) were studied in great detail in the past in [1–5]. Non-symmetric AREs have appeared in [6–9]. For a comprehensive study and applications, one may refer to the recent book by [10], and to a classic paper by [11] and the references therein. In [12] applications of (1) in transport theory are discussed. Non-symmetric ARE also arises in total least squares problem [13,3], spectral factorization of polynomials [6,14]. Spectral factorization of polynomials can be described as factoring a polynomial $p(x)$ as $p(x) = u(x)u(-x)$ (for continuous systems) such that $u(x)$ is either Hurwitz or anti-Hurwitz [6]. One is also interested in the factorization $p(x) = u(x)u(x^{-1})$ (for discrete systems) such that $u(x)$ has all its roots inside the unit circle [14]. These notions of spectral factorization can be naturally extended to rational functions too. It is shown in [6,14] that such factorizations are related to solutions of an algebraic Riccati equation.

The Riccati equation (1) shows up in non-cooperative differential games and stability analysis of $\mathcal{P}$-systems [15].

Definition 1. ([15]) Let $A_1 \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $A_2 \in \mathbb{R}^{l\times l}$, $B_2 \in \mathbb{R}^{l\times m}$, $Q \in \mathbb{R}^{l\times n}$, $L_1 \in \mathbb{R}^{l\times m}$, $L_2 \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{m\times m}$. The continuous-time system defined by Σ:

$$\begin{aligned}
\dot{x} &= A_1 x + B_1 u, x(0) = \xi \\
\dot{\lambda} &= -Qx - A_2^T \lambda - L_1 u \\
v &= L_2^T x + B_2^T \lambda + Ru
\end{aligned}$$

is called a $\mathcal{P}$-system.

Non-symmetric ARE associated with Σ is given by

$$A_2^T X + XA_1 - (XB_1 + L_1)R^{-1}(B_2 X + L_2^T) + Q = 0$$

A solution X to above ARE is said to be right stabilizing if $A_1 + B_1 F_1$ $(F_1 = -R^{-1}(B_2^T X + L_2^T))$ is stable. If $A_2 + B_2 F_2$ is stable for $F_2^T = -(XB_1 + L_1)R^{-1}$, then X is said to be a left stabilizing solution. Therefore, it is interesting to look at eigenvalues of feedback matrices associated with solutions of (1). We explore this in Section 4.

Non-symmetric differential Riccati equation $\dot{K} = AK + KB + C - KDK$ appears in numerical analysis [16]. Its equilibrium points are given by the solutions of ARE. Non-symmetric Algebraic Riccati equation is related to power method and QR algorithm in matrix computations [16,17]. For detailed study of differential Riccati equation, we refer the reader to [18,19].

We are interested in finding all the real solutions of (1). The invariant subspace approach to find solutions of (1) is well known in the literature [3,7,11]. Consider $(n+m)\times$