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Trace and determinant preserving maps of matrices

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We show that if a map ϕ on the set of positive definite matrices satisfies

$$\det(A+B) = \det(\phi(A)+\phi(B)), \qquad \text{or}$$

$$\operatorname{tr}(AB^{-1}) = \operatorname{tr}(\phi(A)\phi(B)^{-1}) \qquad \text{with} \quad \det\phi(I) = 1,$$

then ϕ is of the form $\phi(A) = M^*AM$ or $\phi(A) = M^*A^tM$ for some invertible matrix M with $\det(M^*M) = 1$. We also characterize the map $\phi : \mathcal{S} \to \mathcal{S}$ preserving the similar trace equality or the determinant equality

$$\det(tA+(1-t)B) = \det(t\phi(A)+(1-t)\phi(B)), \qquad t \in [0,1],$$

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in $\mathcal{S}$, where $\mathcal{S}$ denotes the set of complex matrices, symmetric matrices, or upper triangular matrices, respectively.

1. Introduction

Over the last decades, one of the most active research topics in matrix theory is the linear preserver problem [1,11,13,15]. Many interesting results about preserver problems in different matrix algebras were discussed and obtained. They included preservers on determinant, eigenvalue, spectrum, permanent, rank, commutativity, product, trace, norm, etc.

In this paper, we study the relationship between the maps preserving determinants and trace equalities, and characterize this kind of maps. Given $n \in \mathbb{Z}^+$, let M_n (resp., H_n, $\overline{P_n}$, P_n, S_n, T_n, D_n) be the set of $n \times n$ complex (resp., hermitian, positive semi-definite, positive definite, symmetric, upper triangular, diagonal) matrices. For $\mathcal{S} = S_n$, M_n, T_n, or P_n, we show that a map ϕ on $\mathcal{S}$ satisfying trace equalities $\operatorname{tr}(\phi(A)\phi(B)^{-1}) = \operatorname{tr}(AB^{-1})$ or $\operatorname{tr}(\phi(A)\phi(B)) = \operatorname{tr}(AB)$ is linear (Lemmas 2.3 and 2.4). Then we use the result to show that

(1) A map $\phi : P_n \to P_n$ satisfying

$$\det(\phi(A) + \phi(B)) = \det(A + B), \qquad A, B \in P_n \tag{1.1}$$

or $\operatorname{tr}(\phi(A)\phi(B)^{-1}) = \operatorname{tr}(AB^{-1})$ with $\det \phi(I) = 1$ is of the form $\phi(A) = M^*AM$ or $\phi(A) = M^*A^tM$ for some invertible matrix M with $\det(M^*M) = 1$ (Theorem 3.1). Here and throughout this paper A^t denotes the transpose of A.

(2) For $\mathcal{S} = S_n$, M_n, or T_n, and the map $\phi : \mathcal{S} \to \mathcal{S}$, the trace equality $\operatorname{tr}(\phi(A)\phi(B)^{-1}) = \operatorname{tr}(AB^{-1})$ with $\det \phi(I) = 1$ is equivalent to the determinant equality $\det(t\phi(A) + (1-t)\phi(B)) = \det(tA + (1-t)B)$ for all $t \in [0, 1]$ which leads to the explicit forms of ϕ (Theorems 4.1, 5.1, 6.1). This enriches Dolinar, Šemrl, Tan, Wang, Cao, and Tang's results [2,4,16].

The original determinant preserving problem came from Frobenius [6] in 1897. Let $\phi : M_n \to M_n$ be a linear map satisfying

$$\det(\phi(A)) = \det(A), \qquad A \in M_n. \tag{1.2}$$

Then there exist M, $N \in M_n$ with $\det(MN) = 1$ such that either

$$\phi(A) = MAN, \qquad A \in M_n, \tag{1.3}$$