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Rank in Banach algebras: A generalized Cayley–Hamilton theorem

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ABSTRACT

Let A be a semisimple Banach algebra with non-trivial, and possibly infinite-dimensional socle. Addressing a problem raised in [5, p. 1399], we first define a characteristic polynomial for elements belonging to the socle, and we then show that a Generalized Cayley–Hamilton Theorem holds for the associated polynomial. The key arguments leading to the main result follow from the observation that a purely spectral approach to the theory of the socle carries alongside it an efficient method of dealing with relativistic problems associated with infinite-dimensional socles.

1. The characteristic polynomial

Let A be a complex, semisimple Banach algebra with identity element $\mathbf{1}$ and invertible group A^{-1}. For $x \in A$ denote $\sigma_A(x) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin A^{-1}\}$, and $\sigma'_A(x) := \sigma_A(x)\backslash\{0\}$. If the underlying algebra is clear from the context, then we shall agree to omit the subscript A in the notation $\sigma_A(x)$ and $\sigma'_A(x)$. This convention will also be followed in

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the forthcoming definitions of rank, trace, determinant, etc. As in [5], following Aupetit and Mouton in [2], we define the *rank* of $a \in A$ by

$$\operatorname{rank}_A(a) = \sup_{x \in A} \#\sigma'(xa) \leq \infty, \tag{1.1}$$

where the symbol $\#K$ denotes the number of distinct elements in a set $K \subseteq \mathbb{C}$. It can be shown [2, Corollary 2.9] that the socle, written $\operatorname{soc}(A)$, of a semisimple Banach algebra A coincides with the collection $\mathcal{F} := \{a \in A : \operatorname{rank}(a) < \infty\}$ of finite-rank elements. With respect to (1.1) it is further useful to know that $\sigma'(xa) = \sigma'(ax)$ (Jacobson's Lemma). If $x \in A$ is such that $\#\sigma'(xa) = \operatorname{rank}(a)$, then we say *a assumes its rank at x*. An important fact in this regard is that, for each $a \in \operatorname{soc}(A)$, the set

$$E_A(a) = \{x \in A : \#\sigma'(xa) = \operatorname{rank}(a)\} \tag{1.2}$$

is dense and open in A [2, Theorem 2.2]. If $a \in \operatorname{soc}(A)$ assumes its rank at $\mathbf{1}$ then a is said to be a *maximal finite-rank element*. Maximal finite-rank elements are important because they can be "diagonalized" [2, Theorem 2.8]. That is, if $a \in \operatorname{soc}(A)$ satisfies $\operatorname{rank}(a) = \#\sigma'(a) = n$, then a can be expressed as

$$a = \lambda_1 p_1 + \cdots + \lambda_n p_n,$$

where: the λ_i are the distinct nonzero spectral values of a; and the p_i the corresponding Riesz projections, all of which are minimal (and hence rank one). Furthermore, the collection of maximal finite-rank elements is dense in $\operatorname{soc}(A)$.

For $a \in \operatorname{soc}(A)$, Aupetit and Mouton now use the "spectral rank" in (1.1) to define the *trace* and *determinant* as:

$$\operatorname{tr}_A(a) = \sum_{\lambda \in \sigma(a)} \lambda\, m(\lambda, a) \tag{1.3}$$

$$\det_A(a + \mathbf{1}) = \prod_{\lambda \in \sigma(a)} (\lambda + 1)^{m(\lambda, a)} \tag{1.4}$$

where $m(\lambda, a)$ is the *multiplicity of a at* λ. A brief description of the notion of multiplicity in the abstract case goes as follows (for particular details one should consult [2]): Let $a \in \operatorname{soc}(A)$, $\lambda \in \sigma(a)$ and let V_λ be an open disk centered at λ such that V_λ contains no other points of $\sigma(a)$. In [2, pp. 119–120] it is shown that there exists an open ball, say $U \subset A$, centered at $\mathbf{1}$ such that $\#\left[\sigma(xa) \cap V_\lambda\right]$ is constant as x runs through $E(a) \cap U$. This constant integer is the multiplicity of a at λ. If $\lambda \neq 0$ then one can moreover prove that $m(\lambda, a)$ is the rank of the Riesz projection associated to the pair (λ, a). If a is a maximal finite-rank element then $m(\lambda, a) = 1$ [2, p. 120].

In the operator case, $A = \mathcal{L}(X)$, where X is a Banach space, the formulas in (1.1), (1.3), and (1.4) can be shown to coincide with the respective classical operator definitions.