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Almost-invariant and essentially-invariant halfspaces ☆

Gleb Sirotkin *, Ben Wallis

Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, United States

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ABSTRACT

In this paper we study sufficient conditions for an operator to have an almost-invariant half-space. As a consequence, we show that if X is an infinite-dimensional complex Banach space then every operator $T \in \mathcal{L}(X)$ admits an essentially-invariant half-space. We also show that whenever a closed algebra of operators possesses a common AIHS, then it has a common invariant half-space as well.
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* Corresponding author.
E-mail addresses: gsirotkin@niu.edu (G. Sirotkin), bwallis@niu.edu (B. Wallis).

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1. Introduction and the first result

The invariant subspace problem in its full generality was famously closed in the negative by Enflo in a series of papers beginning with [7] and ending with [8], where he constructed an operator acting on a separable Banach space which fails to admit any nontrivial invariant subspace, that is, invariant subspace which is nonzero, proper, and closed. Independently Charles Read produced an outstanding series of examples [17–21], in particular, an operator acting on ℓ_1 [18] which fails to admit any nontrivial invariant subspace. Thus, the answer is still negative even for "nice" Banach spaces. Current work on the invariant subspace problem now focuses on the separable Hilbert space case, which remains unsolved for the time being.

A related but independent problem was posed in [4]. Let us say that a subspace Y of a Banach space X is **almost-invariant** under T whenever $TY \subseteq Y + E$ for some finite-dimensional **error** subspace E. In this case, the smallest possible dimension of E we call the **defect**. To make things nontrivial, we restrict our attention to the case where Y is a **halfspace**, that is, a closed subspace with both infinite dimension and infinite codimension in X. Let us use the abbreviations **AIHS** for "almost-invariant halfspace" and **IHS** for "invariant halfspace." Then we can ask, does every operator on an infinite-dimensional Banach space admit an AIHS? Let us call this the **AIHS problem**.

It turns out that for closed algebras of operators existence of a common AIHS and IHS is equivalent. In [14, Theorem 2.3], this was proved for the case when AIHS is complemented, but the proof can be adapted to the general case as well. Let us demonstrate this. Note also that, although our other results are validated only for complex Banach spaces, this next theorem works even for real Banach spaces.

Theorem 1.1. *Let X be a (real or complex) Banach space, and let $\mathcal{A}$ be a norm-closed algebra of operators in $\mathcal{L}(X)$. If there exists a halfspace Y of X which is almost-invariant under every $A \in \mathcal{A}$, then there exists a halfspace Z which is invariant under every $A \in \mathcal{A}$.*

Proof. Let us assume, without loss of generality, that $\mathcal{A}$ is a unital algebra satisfying the hypothesis of the theorem. By a result of Popov ([15, Theorem 2.7]), there is number $M \in \mathbb{N}$ such that for every $A \in \mathcal{A}$ we have $AY \subseteq Y + F_A$ with $\dim(F_A) \leq M$.

For each $A \in \mathcal{A}$, define $J_A \in \mathcal{L}(Y, X/Y)$ by $J_A := qA|_Y$, where $q : X \to X/Y$ is the canonical quotient map. Let us also assume that there are vectors $y_1, \dots, y_M \in Y$ and that there is $T \in \mathcal{A}$ such that for $W := J_T([y_i]_{i=1}^M)$ we have $\dim(W) = M$. It was proved in [15, Lemma 3.4] that if Y is an AIHS under T and $V := Y \cap T^{-1}(Y)$ then the defect of Y under T is precisely $\dim(Y/V)$. Thus, $\dim(Y/V) = M$. Notice that $V = \mathcal{N}(J_T)$ (the null space of J_T) and $Y = V \oplus [y_i]_{i=1}^M$.

Fix any $A \in \mathcal{A}$ and any vector $v \in V$ and let us show that $J_A v \in W$ holds. Indeed, if not, let $\delta > 0$ be the minimum of $\| \sum_{i=1}^M \alpha_i J_T y_i + \alpha_{M+1} J_A v\|$ over the set of all $(\alpha_i)_{i=1}^{M+1}$ from the unit sphere of $\mathbb{K}^{M+1}$. The minimum is attained due to the com-

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