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Some matrix completions over integral domains

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ABSTRACT

We characterize 3×3 nilpotent matrices which are completions of 2×2 arbitrary matrices and 3×3 idempotent matrices which are completions of 2×2 arbitrary matrices over integral domains. As an application we show that a nil-clean element of a ring which belongs to a corner of the ring, may not be nil-clean in this corner.

1. Introduction

Throughout the last decades, numerous results have been published in the area of the so-called Matrix Completion Problems (see [1] for a recent survey).

In this paper we discuss two such completions over arbitrary (commutative) integral domains. While nilpotents and idempotents can be easily characterized in $\mathcal{M}_2(R)$ for any commutative ring R, it is much harder to do this in $\mathcal{M}_3(R)$. In this short note we characterize nilpotent 3×3 matrices obtained by completing an arbitrary 2×2 matrix

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and idempotent 3×3 matrices obtained by completing an arbitrary 2×2 matrix. As an application we give *a negative example related to* a long lasting question on nil-clean rings, stated by Diesl already in his Ph.D. thesis (2006), and restated in [2]: are corners of nil-clean rings also nil-clean?

More precisely, since so far, this question turns out to be much harder to answer, one may begin by asking more generally (for a ring R and an idempotent $e \in R$) how the nil-clean elements of eRe (denoted in the sequel $NC(eRe)$) are related to those of R (denoted $NC(R)$). If it were true that $eRe \cap NC(R) \subseteq NC(eRe)$ (for any full idempotent $e \in R$), then certainly the question above would have a "yes" answer. However, this inclusion relation does not hold in general, as our example shows.

In this section we present a method of constructing 3×3 completions of 2×2 matrices which are nilpotent respectively idempotent. We describe this construction for matrices over any (commutative) integral domain.

First recall the following formula (folklore): let A and B be square matrices of the same size. Then the trace

$$\mathrm{Tr}(AB) = \sum A * B^T$$

where the RHS is obtained by adding the elements of the elementwise product (*) of the matrices (B^T denotes the transpose).

Next note that for an arbitrary 2×2 matrix M, $\mathrm{Tr}(M^2) = \mathrm{Tr}(M)^2 - 2\det(M)$.

Finally, the characteristic polynomial of a 3×3 matrix is $p_A(X) = \det(X.I_3 - A) = X^3 - \mathrm{Tr}(A)X^2 + \frac{1}{2}(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2))X - \det(A)$. Hence a 3×3 matrix A is nilpotent iff $p_A(X) = X^3$ iff $\det(A) = \mathrm{Tr}(A) = \mathrm{Tr}(A^2) = 0$ in any (commutative) integral domain.

In the sequel, for any given matrix U, u_{ij} denotes the (i, j) entry of U.

Proposition 1. *Let R be a (commutative) integral domain and let U be an arbitrary matrix in $\mathcal{M}_2(R)$. There is a nilpotent matrix $N \in \mathcal{M}_3(R)$ which has U as the northwest 2×2 corner, whenever there exist elements $a, b, x, y \in R$ such that $ax + by = \det(U) - \mathrm{Tr}(U)^2$ and $bxu_{12} + ayu_{21} - axu_{22} - byu_{11} = \mathrm{Tr}(U)\det(U)$. Such a matrix exists if (e.g.) u_{12} or u_{21} is a unit.*

Conversely, if N is a 3×3 nilpotent matrix which has U as the northwest 2×2 corner, the previous relations hold for $a = n_{13}, b = n_{23}, x = n_{31}$ and $y = n_{32}$.

Proof. To simplify the writing we use block multiplication. We search for $N = \begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}$ where $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, $\alpha = \begin{bmatrix} a \\ b \end{bmatrix}$ is a column, $\beta = \begin{bmatrix} x & y \end{bmatrix}$ is a row and $t = \mathrm{Tr}(U) = u_{11} + u_{22}$. Notice that already $\mathrm{Tr}(N) = 0$.