Linear Algebra and its Applications 507 (2016) 462–473

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On graphs with just three distinct eigenvalues

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ARTICLE INFO

Article history:
Received 6 November 2015
Accepted 17 June 2016
Available online 21 June 2016
Submitted by R. Brualdi

MSC:
05C50

Keywords:
Main eigenvalue
Minimum degree
Strongly regular graph
Symmetric 2-design
Vertex-deleted subgraph

ABSTRACT

Let G be a connected non-bipartite graph with exactly three distinct eigenvalues ρ, μ, λ, where $\rho > \mu > \lambda$. In the case that G has just one non-main eigenvalue, we find necessary and sufficient spectral conditions on a vertex-deleted subgraph of G for G to be the cone over a strongly regular graph. Secondly, we determine the structure of G when just μ is non-main and the minimum degree of G is $1 + \mu - \lambda\mu$: such a graph is a cone over a strongly regular graph, or a graph derived from a symmetric 2-design, or a graph of one further type.

1. Introduction

Let G be a graph of order n with $(0, 1)$-adjacency matrix A. An eigenvalue σ of A is said to be an eigenvalue of G, and σ is a *main* eigenvalue if the eigenspace $\mathcal{E}_A(\sigma)$ is not orthogonal to the all-1 vector in $\mathbb{R}^n$. Always the largest eigenvalue, or *index*, of G is a main eigenvalue, and it is the only main eigenvalue if and only if G is regular. We say that G is an *integral* graph if every eigenvalue of G is an integer. We use the

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http://dx.doi.org/10.1016/j.laa.2016.06.031

notation of the monograph [5], where the basic properties of graph spectra can be found in Chapter 1.

Let $\mathcal{C}_1$ be the class of connected graphs with just three distinct eigenvalues, and let $\mathcal{C}_2$ be the class of connected graphs with exactly two main eigenvalues. It is an open problem to determine all the graphs in $\mathcal{C}_1$, and another open problem to determine all the graphs in $\mathcal{C}_2$. Here we investigate graphs in $\mathcal{C}_1 \cap \mathcal{C}_2$. From [6, Propositions 2 and 3] we know that if G is a non-integral graph in $\mathcal{C}_1$ then either G is complete bipartite or the two smaller eigenvalues of G are algebraic conjugates. In the latter case, G has exactly 1 or 3 main eigenvalues, and so a graph in $\mathcal{C}_1 \cap \mathcal{C}_2$ is either integral or complete bipartite.

The class $\mathcal{C}_1$ contains all connected non-complete strongly regular graphs; moreover it is known that if H is a strongly regular graph of order n with eigenvalues $\nu > \mu > \lambda$ then the cone $K_1 \triangledown H$ lies in $\mathcal{C}_1$ if and only if $\lambda(\nu - \lambda) = -n$ (see [8] and Lemma 2.1 below). We shall see in Section 2 that the condition $\lambda(\nu - \lambda) = -n$ is equivalent to the condition $\nu = \mu(1 - \lambda)$, and that when this condition is satisfied we have $K_1 \triangledown H \in \mathcal{C}_1 \cap \mathcal{C}_2$. There are infinitely many strongly regular graphs which satisfy the condition (see [8, Proposition 7.1]); examples include the Petersen graph ($\mu = 1,\ \lambda = -2$), the Gewirtz graph ($\mu = 2,\ \lambda = -4$) and the Chang graphs ($\mu = 4,\ \lambda = -2$).

Now let G be a non-bipartite graph in $\mathcal{C}_1 \cap \mathcal{C}_2$ with spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$ where $\rho > \mu > \lambda$. In Section 3, we prove that the following are equivalent: (a) G is the cone over a strongly regular graph, (b) G has a vertex-deleted subgraph with just three distinct eigenvalues, (c) G has a vertex-deleted subgraph with index $\nu = \mu(1 - \lambda)$. In particular, for $G \in \mathcal{C}_1 \cap \mathcal{C}_2$, application of the condition $\nu = \mu(1 - \lambda)$ is not confined to a strongly regular graph H such that $G = K_1 \triangledown H$.

We note that $\mathcal{C}_1 \cap \mathcal{C}_2$ also contains the graphs constructed by van Dam [6] from a symmetric 2-$(q^3 - q + 1, q^2, q)$ design $\mathcal{D}$: such a graph is obtained from the incidence graph of $\mathcal{D}$ by adding an edge between each pair of blocks. We refer to such graphs as graphs of *symmetric type*; they exist whenever q is a prime power and there exists a projective plane of order $q - 1$ [7]. Their eigenvalues are q^3, $q - 1$, $-q$ with multiplicities 1, $q^3 - q$, $q^3 + 1$ respectively. These graphs share with the cones described above the properties that μ is non-main and $1 + \mu - \mu\lambda = \delta(G)$, the minimum degree in G. In Section 4, we determine the structure of all graphs in $\mathcal{C}_1 \cap \mathcal{C}_2$ with these properties.

2. Preliminaries

Our first proof begins with a short derivation of the condition $\lambda(\nu - \lambda) = -n$, which was obtained by other means in [8, Proposition 6.1(b)].

Lemma 2.1. *Let H be a strongly regular graph of order n with spectrum $\nu, \mu^{(s)}, \lambda^{(t)}$, where $\nu > \mu > \lambda$. Then $K_1 \triangledown H$ has just three distinct eigenvalues if and only if $\lambda(\nu - \lambda) = -n$, equivalently $\nu = \mu(1 - \lambda)$. In this situation, $K_1 \triangledown H$ has spectrum $\rho, \mu^{(s)}, \lambda^{(t+1)}$, where $\rho = \nu - \lambda$, and the main eigenvalues of $K_1 \triangledown H$ are ρ and λ.*