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Spaces of matrices of constant rank and uniform vector bundles

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ABSTRACT

We consider the problem of determining $l(r,a)$, the maximal dimension of a subspace of $a \times a$ matrices of rank r. We first review, in the language of vector bundles, the known results. Then using known facts on uniform bundles we prove some new results and make a conjecture. Finally we determine $l(r;a)$ for every r, $1 \leq r \leq a$, when $a \leq 10$, showing that our conjecture holds true in this range.

0. Introduction

Let A, B be k-vector spaces of dimensions a, b (k algebraically closed, of characteristic zero). A sub-vector space $M \subset \mathcal{L}(A,B)$ is said to be of (constant) rank r if every $f \in M, f \neq 0$, has rank r. The question considered in this paper is to determine

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$l(r,a,b) := max\{\dim M \mid M \subset \mathcal{L}(A,B) \text{ has rank } r\}$. This problem has been studied some time ago by various authors [22,20,4,9] and has been recently reconsidered, especially in its (skew) symmetric version [17,18,16,5].

It is known, at least since [20], that to give a subspace M of constant rank r, dimension $n+1$, is equivalent to give an exact sequence: $0 \to F \to a.\mathcal{O}(-1) \stackrel{\psi}{\to} b.\mathcal{O} \to E \to 0$, on $\mathbb{P}^n$, where F, E are vector bundles of ranks $(a-r)$, $(b-r)$. Our starting point is to observe that the bundle $\mathcal{E} := Im(\psi)$, of rank r, is *uniform*, of splitting type $(-1^c, 0^{r-c})$, where $c := c_1(E)$ (Lemma 2). This had been previously observed (but not really exploited) in the cases of spaces of symmetric or skew-symmetric maps [17]. This allows us to apply the known results (and conjectures) on uniform bundles.

This paper is organized as follows. In the first section we recall some basic facts and fix the notations. Then in Section two, we set $a = b$ to fix the ideas and we survey the known results (at least those we are aware of), giving a quick, uniform (!) treatment in the language of vector bundles. In Section three, using known results on uniform bundles, we obtain a new bound on $l(r;a)$ in the range $(2a+2)/3 > r > (a+2)/2$ (as well as some other results, see Theorem 18). By the way we don't expect this bound to be sharp. Indeed by "translating" (see Proposition 17) a long standing conjecture on uniform bundles (Conjecture 1), we conjecture that $l(r;a) = a-r+1$ in this range (see Conjecture 2). Finally, with some ad hoc arguments, we show in the last section, that our conjecture holds true for $a \leq 10$ (actually we determine $l(r;a)$ for every r, $1 \leq r \leq a$, when $a \leq 10$).

1. Generalities

Following [20], to give $M \subset Hom(A,B)$, a sub-space of constant rank r, with $\dim(M) = n+1$, is equivalent to give on $\mathbb{P}^n$, an exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_M & \longrightarrow & a.\mathcal{O}(-1) & \xrightarrow{\psi_M} & b.\mathcal{O} & \longrightarrow & E_M \longrightarrow 0 \\ & & & & \searrow & & \nearrow & & \\ & & & & & \mathcal{E}_M & & & \end{array} \quad (1)$$

where $\mathcal{E}_M = Im(\psi_M), F_M, E_M$ are vector bundles of ranks $r, a-r, b-r$ (in the sequel we will drop the index M if no confusion can arise).

Indeed the inclusion $i : M \hookrightarrow Hom(A,B)$ is an element of $Hom(M, A^\vee \otimes B) \simeq M^\vee \otimes A^\vee \otimes B$ and can be seen as a morphism $\psi : A \otimes \mathcal{O} \to B \otimes \mathcal{O}(1)$ on $\mathbb{P}(M)$ (here $\mathbb{P}(M)$ is the projective space of lines of M). At every point of $\mathbb{P}(M)$, ψ has rank r, so the image, the kernel and the cokernel of ψ are vector bundles.

A different (but equivalent) description goes as follows: we can define $\psi : A \otimes \mathcal{O}(-1) \to B \otimes \mathcal{O}$ on $\mathbb{P}(M)$, by $v \otimes \lambda f \to \lambda f(v)$.

The vector bundle $\mathcal{E}_M$ is of a particular type.