Linear Algebra and its Applications 507 (2016) 499–512

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A cospectral family of graphs for the normalized Laplacian found by toggling

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ARTICLE INFO

Article history:
Received 23 March 2016
Accepted 21 June 2016
Available online 23 June 2016
Submitted by R. Brualdi

MSC:
05C50

Keywords:
Cospectral graphs
Normalized Laplacian
Toggling
Characteristic polynomial

ABSTRACT

We give a construction of a family of (weighted) graphs that are pairwise cospectral with respect to the normalized Laplacian matrix, or equivalently, probability transition matrix. This construction can be used to form pairs of cospectral graphs with different number of edges, including situations where one graph is a subgraph of the other. The method used to demonstrate cospectrality is by showing the characteristic polynomials are equal.

1. Introduction

Spectral graph theory studies the relationship between the structure of a graph and the eigenvalues of a particular matrix associated with that graph. There are several matrices that are commonly studied, each with merits and limitations. These limitations exist because graphs can be constructed which have the same spectrum with respect to the matrix and are fundamentally different in some structural aspect. Such graphs are called *cospectral*.

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1 Partially supported by an NSA Young Investigator Grant.

http://dx.doi.org/10.1016/j.laa.2016.06.033
0024-3795/

There are many possible matrices to consider, and the matrix we consider in this paper is the normalized Laplacian (see [3,6]). The rows and columns of this matrix are indexed by the vertices, and for a simple graph the matrix is defined as follows:

$$\mathcal{L}(i,j) = \begin{cases} 1 & \text{if } i=j\text{, and vertex } i \text{ is not isolated;} \\ \dfrac{-1}{\sqrt{d_i d_j}} & \text{if } i{\sim}j; \\ 0 & \text{otherwise;} \end{cases}$$

where d_i is the degree of vertex i.

In this paper we want to look at the more general setting of edge-weighted graphs, i.e., there is a symmetric, non-negative weight function, $w(i,j)$ on the edges. The degree of a vertex now corresponds to the sum of the weights of the incident edges, i.e., $d_i = \sum_{i\sim j} w(i,j)$. The normalized Laplacian for weighted graphs is defined in the following way:

$$\mathcal{L}(i,j) = \begin{cases} 1 & \text{if } i=j\text{, and vertex } i \text{ is not isolated;} \\ \dfrac{-w(i,j)}{\sqrt{d_i d_j}} & \text{if } i{\sim}j; \\ 0 & \text{otherwise.} \end{cases}$$

(A simple graph corresponds to the case where $w(i,j) \in \{0,1\}$ for all i,j.) We note that when the graph has no isolated vertices, $\mathcal{L}$ can be written as $\mathcal{L} = D^{-1/2}(D-A)D^{-1/2}$, where $A_{i,j} = w(i,j)$ and D is the diagonal degree matrix. Finally, we point out that this matrix is connected with the probability transition matrix $D^{-1}A$ of a random walk. In particular, two graphs with no isolated vertices are cospectral for $\mathcal{L}$ if and only if they are cospectral for $D^{-1}A$.

There has been some interest in the construction of cospectral graphs for the normalized Laplacian. Cavers [5] showed that a restricted variation of Godsil–McKay switching (see [8]) preserves the spectrum, while Butler and Grout [4] showed that gluing in two different special bipartite graphs into some arbitrary graph resulted in a pair of cospectral graphs. In both cases, the operation preserved the number of edges in the graph.

On the other hand, it is possible for graphs with different number of edges to be cospectral with respect to the normalized Laplacian. The classical example of this is complete bipartite graphs $K_{p,q}$ which have spectrum $\{0, 1^{(p+q-2)}, 2\}$ (here the exponent is indicating multiplicity). For example, the (sparse) star $K_{1,2n-1}$ is cospectral with the (dense) regular graph $K_{n,n}$. Until recently, this was the *only* known construction of cospectral graphs with different number of edges. Butler and Grout [4] gave some examples of small graphs found by exhaustive computation that differ in the number of edges, including some where one graph was a subgraph of the other. Butler [2] expanded