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The asymptotic leading term for maximum rank of ternary forms of a given degree

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A B S T R A C T

Let $\mathrm{r}_{\max}(n, d)$ be the maximum Waring rank for the set of *all* homogeneous polynomials of degree $d > 0$ in n indeterminates with coefficients in an algebraically closed field of characteristic zero. To our knowledge, when $n, d \geq 3$, the value of $\mathrm{r}_{\max}(n, d)$ is known only for $(n, d) = (3, 3), (3, 4), (3, 5), (4, 3)$. We prove that $\mathrm{r}_{\max}(3, d) = d^2/4 + O(d)$ as a consequence of the upper bound $\mathrm{r}_{\max}(3, d) \leq \lfloor (d^2 + 6d + 1)/4 \rfloor$.

1. Introduction

A natural kind of Waring problem asks for the least of the numbers r such that *every* homogeneous polynomial of degree $d > 0$ in n indeterminates can be written as a sum of r dth powers of linear forms. For instance, when $(n, d) = (3, 4)$ (and the coefficients are taken in an algebraically closed field of characteristic zero), the answer is 7. This was

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found for the first time in [11]. In view of the interplay with the rank of tensors, relevant applicative interests of questions like this have recently been recognized (see [12]). For further information we refer the reader to [13, Introduction].

Every power sum decomposition gives rise to a set of points in the projectivized space of linear forms, and in [8] it is proved that for ternary quartics one can always obtain a power sum decomposition by considering seven points arranged along three suitably predetermined lines. In [9], considering sets of points arranged along four lines, one finds that every ternary quintic is a sum of 10 fifth powers of linear forms. Ternary quintics without power sum decompositions with less than 10 summands were exhibited soon after in [5]. Hence, the answer in the case $(n, d) = (3, 5)$ is 10.

In the present paper we test "at infinity" the technique of arranging decompositions of ternary forms along suitably predetermined lines. More precisely, let $\mathrm{r}_{\max}(n, d)$ denote the desired answer to the mentioned Waring problem. For each n, since the dth powers of linear forms span the whole vector space of degree d forms, $\mathrm{r}_{\max}(n, d)$ is bounded above by the dimension of that space:

$$\mathrm{r}_{\max}(n, d) \leq \binom{d+n-1}{n-1}. \tag{1}$$

On the other hand, the set of all sums of r dth powers of linear forms can not cover that space when $rn < \binom{d+n-1}{n-1}$, by dimension reasons; hence

$$\mathrm{r}_{\max}(n, d) \geq \frac{1}{n}\binom{d+n-1}{n-1}. \tag{2}$$

Let us also recall that $rn - 1$ is the *expected* dimension of the r-th secant variety $\sigma_r\left(V_{n-1,d}\right)$ of the dth $(n-1)$-dimensional Veronese variety $V_{n-1,d}$ in $\mathbb{P}^N$, with $N := \binom{d+n-1}{n-1} - 1$. Hence, if one looks at the above statement from a geometric viewpoint, it amounts to saying that when $rn < \binom{d+n-1}{n-1}$ it must be $\sigma_k\left(V_{n-1,d}\right) \subsetneq \mathbb{P}^N$. To see this, the obvious fact that $\dim \sigma_k\left(V_{n-1,d}\right)$ can not exceed the expected dimension must be taken into account. It is also worth mentioning that, as a matter of fact, $\sigma_r\left(V_{n-1,d}\right)$ is actually of the expected dimension, except for a small list of values of (r, n, d) which is completely known. That is a difficult and important result due to Alexander and Hirschowitz (see [1]). In particular, it gives the solution of the "generic" version of the Waring problem we are dealing with. That is, it gives the least of the numbers r such that *generic* (i.e., almost all) homogeneous polynomials of degree $d > 0$ in n indeterminates can be written as a sum of r dth powers of linear forms.

The bounds (1) and (2) show that for each fixed n we have $\mathrm{r}_{\max}(n, d) = O\left(d^{n-1}\right)$, and if $\mathrm{r}_{\max}(n, d) = c_n d^{n-1} + O\left(d^{n-2}\right)$ for some constant c_n (as it is reasonable to expect), then it must be $1/n! \leq c_n \leq 1/(n-1)!$. The best general upper bound on $\mathrm{r}_{\max}(n, d)$ to our knowledge is given by [3, Corollary 9]. This implies that the constant c_n (if it exists) is at most $2/n!$. Using [6, Proposition 4.1] (see also [4, Theorem 7], [5, Theorem 1]), we deduce $\mathrm{r}_{\max}(3, d) \geq \left\lfloor (d+1)^2/4 \right\rfloor$. Hence, it must be $1/4 \leq c_3 \leq 1/3$. In the present work,