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Explicit formulae for limit periodic flows on networks

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In recent papers it was shown that, under certain conditions, the C_0-semigroup describing a flow on a network (metric graph) that contains terminal strong (ergodic) components converges to the direct sum of periodic semigroups generated by the flows on these components. In this note we shall provide an explicit description of these limit semigroups in terms of the components of the adjacency matrix of the line graph of the network. The result is based on the Frobenius–Perron theory and the estimates of long term behaviour of iterates of reducible matrices.

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1. Introduction

Dynamical problems on networks have been considered recently in many papers, see e.g. [1–3,5,6,9,13,15] and also the monograph [16]. Here we focus on transport on networks, modelled by a system of first order transport equations on the edges of the digraph representing the network. The equations are coupled by Kirchhoff's conditions at the vertices of the diagraph. The problem lends itself to a semigroup theoretical approach which, in particular, allows for proving elegant results on the long term behaviour of the flow on the network, see e.g. [1,9,14,17]. The main result of these papers is that, under certain assumptions on the structure of the network and on the speeds of the flow along each edge, the flow is asymptotically periodic. More precisely, the semigroup generated by the flow converges exponentially to the direct sum of periodic semigroups on strong terminal (ergodic) components of the network. While theoretically very interesting, such a result is somehow incomplete from the practical point of view as it does not provide any explicit formulae for the limit semigroups. In other words, to be able to use the limit periodic semigroups as an approximation of the original flow, one needs to provide the appropriate initial conditions for them and this requires precisely knowing how the material from the transient parts enters the ergodic part of the network.

In this paper we complete the theory by deriving the explicit formulae for the action of the limit semigroups by expressing them by the coefficients of the weighted adjacency matrix of the line graph of the original network.

The approach is based on the proof of asymptotic periodicity given in [1] that uses the representation of the flow as the composition of the iterates of the adjacency matrix of the line graph and the solutions of the transport equations on the edges. Then limit periodic semigroups are then defined by the spectral decomposition of this matrix. If the adjacency matrix in question is primitive, then the relation between its iterates and its spectral decomposition is well-known. However, in general, the adjacency matrix related to the flow is reducible with imprimitive irreducible components and its spectral decomposition becomes too complicated for practical purposes. Though for such matrices there is a well-developed theory of Cesàro limits, see e.g. [19, Section 8.4] or [12, Section 8.6], it only provides information about the averaged behaviour of the iterates and thus it is not immediately useful to analyse the finer details of the limit periodic structure of the flow.

We observe that the structure of spectra of imprimitive reducible matrices recently has come under scrutiny, e.g. in [18], but the goals of the [18] are different from ours. To the author's knowledge there have been few papers dealing with iterates of reducible or imprimitive matrices. In [8] the author focused on reducible matrices with primitive diagonal blocks. The reference [20] specifically deals with Leslie matrices which might be imprimitive and, in Section 3, the author uses an idea that is similar to ours but stops short of finding the formulae for the limit dynamics. None of these results can be directly applied to qualitatively describe the limit periodic semigroups of a network flow which is the main aim of this note.