Linear Algebra and its Applications 508 (2016) 14–21

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the similarity of AB and BA for normal and other matrices [☆]

Stephan Ramon Garcia [a], David Sherman [b,*], Gary Weiss [c]

[a] Department of Mathematics, Pomona College, 610 N. College Ave., Claremont, CA 91711, United States
[b] Department of Mathematics, University of Virginia, P.O. Box 400137, Charlottesville, VA 22904-4137, United States
[c] Department of Mathematical Sciences, 4199 French Hall West, University of Cincinnati, 2815 Commons Way, Cincinnati, OH 45221-0025, United States

ARTICLE INFO

Article history:
Received 14 May 2016
Accepted 28 June 2016
Available online 1 July 2016
Submitted by R.A. Horn

MSC:
15A03
15A18

Keywords:
Matrix similarity

ABSTRACT

It is known that AB and BA are similar when A and B are Hermitian matrices. In this note we answer a question of F. Zhang by demonstrating that similarity can fail if A is Hermitian and B is normal. Perhaps surprisingly, similarity does hold when A is positive semidefinite and B is normal.

☆ The authors would like to acknowledge the support of NSF grants DMS-1265973 and DMS-1201454, and of Simons Collaboration Grant for Mathematicians 245014.

* Corresponding author.

E-mail addresses: stephan.garcia@pomona.edu (S.R. Garcia), dsherman@virginia.edu (D. Sherman), gary.weiss@uc.edu (G. Weiss).

URLs: http://pages.pomona.edu/~sg064747 (S.R. Garcia), http://people.virginia.edu/~des5e (D. Sherman), http://math.uc.edu/~weiss (G. Weiss).

http://dx.doi.org/10.1016/j.laa.2016.06.044
0024-3795/

1. Introduction

Throughout this paper A and B denote complex square matrices of the same size. We pursue the following question: when is AB similar to BA?

This does not always happen. But it does when A and B are Hermitian or when either is invertible; we seek other assumptions that imply similarity. For instance, it was asked by F. Zhang (personal communication) whether it suffices for A and B to be merely normal. We show here that similarity does not follow even when A is Hermitian and B is normal (Example 5.3), although it does if A is further assumed to be positive semidefinite (Theorem 6.1). We also show that similarity, or unitary similarity, follows under various hypotheses when one or both matrices have low rank or size, and we give minimal counterexamples showing that our conditions are sharp.

Similarity will be denoted by $\sim$ and unitary similarity by $\sim_u$.

We thank Fuzhen Zhang for bringing this problem to our attention, and Roger Horn for suggesting significant improvements to Section 6.

2. Ranks of powers of a matrix

We define the *rank sequence* of A to be $\{\operatorname{rank}(A^j)\}_{j=0}^{\infty}$ (with $A^0 = I$). Which sequences of nonnegative integers occur as the rank sequence of a matrix?

Since rank is unchanged by similarity, we may as well consider the Jordan form of A. Jordan blocks for nonzero eigenvalues are invertible, and the ranks of powers of a Jordan block for a zero eigenvalue drop by one until reaching zero. So the drop from $\operatorname{rank}(A^j)$ to $\operatorname{rank}(A^{j+1})$ is precisely the number of Jordan blocks for zero of size at least $j + 1$. The size of these drops is then nonincreasing in j, leading to the conclusion that rank sequences are nonincreasing and convex. We use this fact in Section 5. (Actually it is a characterization of rank sequences, as any nonincreasing convex sequence of nonnegative integers is the rank sequence of a matrix whose Jordan blocks satisfy the criterion just mentioned. Details are left to the interested reader.)

The rank sequence of A carries the same information as the Jordan structure of A for the zero eigenvalue, which is more commonly encoded in the Segre or Weyr characteristic (see [6]), but rank sequences are more natural for this paper.

3. Known facts

If one of the matrices is invertible, then $AB \sim BA$ (conjugate by the invertible one). But even for 2×2 matrices, AB need not be similar to BA: consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$