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Distribution of Laplacian eigenvalues of graphs

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ABSTRACT

Let G be a graph of order n with m edges and clique number ω. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$ be the Laplacian eigenvalues of G and let $\sigma = \sigma(G)$ $(1 \leq \sigma \leq n)$ be the largest positive integer such that $\mu_\sigma \geq \frac{2m}{n}$. In this paper we study the relation between σ and ω. In particular, we provide the answer to Problem 2.3 raised in Pirzada and Ganie (2015) [15]. Moreover, we characterize all connected threshold graphs with $\sigma < \omega - 1$, $\sigma = \omega - 1$ and $\sigma > \omega - 1$. We obtain Nordhaus–Gaddum-type results for σ. Some relations between σ with other graph invariants are obtained.

1. Introduction

In this paper, we consider simple graphs $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. Let $\mathbf{A}(G)$ be the $(0, 1)$-adjacency matrix of G and $\mathbf{D}(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$. This matrix has nonnegative eigenvalues $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The spectrum of $L(G)$ is the Laplacian spectrum of

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G and is denoted by $Lspec(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$. When more than one graph is under consideration, then we write $\mu_i(G)$ instead of μ_i.

For a graph G, consider the number $\sigma = \sigma(G)$ of the Laplacian eigenvalues greater than or equal to the average degree $\overline{d} = \frac{2m}{n}$. More precisely σ is the largest integer for which $\mu_\sigma \geq \frac{2m}{n}$. Here we introduce the parameter σ as a graph invariant.

The main motivation to study this number as a spectral parameter may be the Laplacian energy of a graph G, defined by Gutman and Zhou [8] as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|. \tag{1}$$

Inspired by the concept of graph energy, this parameter has been studied extensively (see, for example, [3–5,8]). For basic properties and extensions of graph energies, see [7,11].

Let σ be the parameter just defined. It is easy to see that (see, [3])

$$LE(G) = 2\mathcal{S}_\sigma(G) - \frac{4m\sigma}{n}, \tag{2}$$

where

$$\mathcal{S}_\sigma(G) = \sum_{i=1}^{\sigma} \mu_i(G).$$

Finding σ and/or bounds for S_σ allows one to obtain bounds for the Laplacian energy. In turn, this may be used to obtain extremal graphs for the Laplacian energy. As an example, in [6], this technique was used to prove that the star $K_{1,n-1}$ is the tree on n vertices with largest Laplacian energy.

It is worth noticing that the value of σ sheds light on the distribution of the Laplacian eigenvalues of a graph G. For example, it is conjectured that most of the Laplacian eigenvalues of trees are smaller than the average degree, e.g. there are a few large eigenvalues while many are small. More precisely the following was conjectured in [18].

Conjecture 1. *Let T be a tree on n vertices, then $\sigma(T) \leq \lfloor \frac{n}{2} \rfloor$.*

The value σ has been used in many papers in different situations. In this paper, we unify information and gather properties of σ as a graph invariant. A second goal of this note is to relate σ with classical parameters. Specifically, we study the relation of σ with the clique number $\omega = \omega(G)$, the number of vertices of the largest complete induced subgraph of G.

Pirzada and Ganie presented the following conjecture in [15]:

Conjecture 2. [15] *Let G be a connected graph with clique number ω. Then $\sigma \geq \omega - 1$.*