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Numerical ranges of tensors

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The main aim of this paper is to generalize matrix numerical ranges to the tensor case based on tensor norms. We show that the basic properties of matrix numerical ranges such as compactness and convexity are valid for tensor numerical ranges. We make use of convexity property to propose an algorithm for approximating tensor numerical ranges in which tensor eigenvalues are contained. Also we consider tensor numerical ranges based on inner products, however, they may not be convex in general.

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1. Introduction

The numerical range (or the field of values) is a set of complex numbers associated with a given n-by-n matrix A:

$$F(A) := \{x^* Ax : x \in \mathbb{C}^n, x^* x = 1\}. \tag{1}$$

One of the fundamental facts concerning the numerical range is, stated by the Toeplitz–Hausdorff theorem (see for instance [10]), that $F(A)$ is a convex subset of $\mathbb{C}$ for any n-by-n matrix A. Another basic and important fact of numerical range is that $F(A)$ contains the spectrum of A. It is known that the numerical range of a square matrix A has an equivalent representation (see [1]):

$$W_{\|\cdot\|_2}(A) = \bigcap_{\lambda \in \mathbb{C}} \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda I\|_2\}, \tag{2}$$

where $\|\cdot\|_2$ is the matrix 2-norm. In other words, the numerical range of a matrix is equivalent to the intersection of a series of closed discs. According to (1) and (2), we see that the numerical range of a matrix can be defined by using a matrix norm or an inner product.

Recently, the definition of a numerical range has been naturally generalized into rectangular matrices by Chorianopoulos et al. [5]. For any $A, B \in \mathbb{C}^{n\times m}$ (n-by-m matrices) and any matrix norm $\|\cdot\|$, the numerical range of A with respect to B defined by

$$\begin{aligned} W_{\|\cdot\|}(A, B) &= \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda B\|, \ \forall \lambda \in \mathbb{C}\} \\ &= \bigcap_{\lambda \in \mathbb{C}} \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda B\|\}, \end{aligned} \tag{3}$$

is a compact and convex set. Obviously, for $n = m$, $B = I$, and $\|\cdot\| = \|\cdot\|_2$, $W_{\|\cdot\|}(A, B)$ is the same as the classical numerical range $F(A)$.

The main purpose of this paper is to generalize matrix numerical ranges to tensor numerical ranges.

1.1. Tensors

A tensor can be represented by a multi-dimensional array. An mth-order n-dimensional tensor can be defined as $\mathcal{A} = (a_{i_1,i_2,\cdots,i_m})$ in which $1 \leq i_1, i_2, \cdots, i_m \leq n$. If for any permutation of $(1, 2, \cdots, m)$, $(j_1, j_2, \cdots, j_m)$, the entries satisfy

$$a_{i_1,i_2,\cdots,i_m} = a_{i_{j_1},i_{j_2},\cdots,i_{j_m}}, \quad 1 \leq i_1, i_2, \cdots, i_m \leq n,$$

then $\mathcal{A}$ is called a symmetric tensor.