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Remarks on the energy of regular graphs

V. Nikiforov

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA

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A B S T R A C T

The energy of a graph is the sum of the absolute values of the eigenvalues of its adjacency matrix. This note is about the energy of regular graphs. It is shown that graphs that are close to regular can be made regular with a negligible change of the energy. Also a k-regular graph can be extended to a k-regular graph of a slightly larger order with almost the same energy. As an application, it is shown that for every sufficiently large n, there exists a regular graph G of order n whose energy $\|G\|_*$ satisfies

$$\|G\|_* > \frac{1}{2}n^{3/2} - n^{13/10}.$$

Several infinite families of graphs with maximal or submaximal energy are given, and the energy of almost all regular graphs is determined.

1. Introduction

In [8] Gutman introduced the energy of a graph as the sum of the absolute values of the eigenvalues of its adjacency matrix. Since the energy of a graph G is the trace norm of its adjacency matrix, we write $\|G\|_*$ for the energy of G.

E-mail address: vnikifrv@memphis.edu.

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In this note we discuss the energy of regular graphs, an area that has been studied (see, e.g., [10] and [9]), but which is still vastly unexplored.

In the groundbreaking paper [12], Koolen and Moulton showed that

$$\|G\|_* \leq k + \sqrt{k(n-k)(n-1)} \tag{1}$$

for every k-regular graph of order n. Equality holds in (1) if and only if $G = K_n$, or $G = (n/2)\,K_2$, or G is a strongly regular graph with parameters (n, k, a, a), i.e., a k-regular design graph (see, e.g., [4], p. 144). Since bound (1) is precise for an amazing variety of graphs, in [3] Balakrishnan proposed to study how tight this bound is in general, and asked the following relevant question, which is quoted here verbatim:

Question 1 *(Balakrishnan [3]). For any two positive integers n and k, $n-1 > k \geq 2$ and $\varepsilon > 0$, does there exist a k-regular graph G with $\|G\|_*/B_2 > 1-\varepsilon$, where $B_2 = k + \sqrt{k(n-1)(n-k)}$.*

Unfortunately, despite its sound idea, Question 1 is incoherent in the above form, because the quantifier of ε is unclear, and if n and k do not depend on ε, the answer is almost always negative. These weaknesses have been exploited in the literature to trivialize Balakrishnan's question, e.g., in [13]. Moreover, important recent results of van Dam, Haemers, and Koolen [6] imply that the original question of Balakrishnan's cannot be preserved without excising many combinations of n and k. To clarify this point, we state an essential corollary of the paper [6], which, however, has eluded its authors:

Theorem 2. *Let $k \geq 2$ and $n \geq k^2 - k + 1$. If G is a k-regular graph of order n, then*

$$\|G\|_* \leq \left(\sqrt{k-1} + \frac{1}{k + \sqrt{k-1}}\right) n. \tag{2}$$

Equality holds in (2) if and only if G is a disjoint union of incidence graphs of projective planes of order $k-1$ or $k = 2$ and G is disjoint union of triangles and hexagons.

Note that for $n > k^2 - k + 1$ the right side of (1) is greater than the right side of (2), so Theorem 2 is a clear improvement over (1). Therefore, if we want to investigate k-regular graphs for which (1) is almost tight, we must suppose that $k \geq \sqrt{n}$.

With this hindsight, it seems natural to split Question 1 into two conjectures: First, study Question 1 for k-regular graphs whenever k grows not too fast with n, say, $k = o(n)$. For this case we venture the following conjecture:

Conjecture 3. *For every $\varepsilon > 0$, there exist $\delta > 0$ and $k_0(\varepsilon)$ such that if $n\delta > k > k_0(\varepsilon)$ and kn is even, there exists a k-regular graph G of order n with*

$$\|G\|_* \geq (1-\varepsilon)\sqrt{k}n.$$