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Generalizations of the Brunn–Minkowski inequality

Juntong Liu

School of Mathematics and Statistics, Fuyang Normal College, Fuyang, 236041, China

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A B S T R A C T

Yuan and Leng (2007) gave a generalization of the matrix form of the Brunn–Minkowski inequality. In this note, we first give a simple proof of this inequality, and then show a generalization of this to a larger class of matrices, namely, matrices whose numerical ranges are contained in a sector.

1. Introduction

The Brunn–Minkowski inequality is one of the most important geometric inequalities. There is a vast amount of work on its generalizations and on its connections with other areas. An excellent survey on this inequality is provided by Gardner (see [5]). The matrix form of the Brunn–Minkowski inequality (e.g., [8, p. 510]) states that if A and B are positive definite matrices of order n, then

$$(\det(A+B))^{1/n} \geq (\det(A))^{1/n} + (\det(B))^{1/n}, \tag{1.1}$$

E-mail address: juntongliu82@163.com.

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with equality if and only if $A = cB(c > 0)$, where $\det(A)$ denotes the determinant of A.

In [4], Ky Fan gave a generalization of the above inequality (1.1). He established the following elegant inequality.

Let A_k denote the kth leading principal sub-matrix of A. If $C = A + B$, where A and B are positive definite matrices of order n, then

$$\left(\frac{\det(C)}{\det(C_k)}\right)^{\frac{1}{n-k}} \geq \left(\frac{\det(A)}{\det(A_k)}\right)^{\frac{1}{n-k}} + \left(\frac{\det(B)}{\det(B_k)}\right)^{\frac{1}{n-k}}. \tag{1.2}$$

In [14], Yuan and Leng gave a generalization of the inequality (1.2). They proved the following result of the matrix form of the Brunn–Minkowski inequality.

Theorem 1.1. *(See [14, Theorem 1.1].) Let A and B be positive definite matrices of order n and let A_k and B_k be the kth leading principal sub-matrix of A and B, respectively. Let a and b be two nonnegative real numbers such that $A > aI_n$ and $B > bI_n$. If $C = A + B$, then*

$$\left(\frac{\det(C)}{\det(C_k)} - \det((a+b)I_{n-k})\right)^{\frac{1}{n-k}} \geq \left(\frac{\det(A)}{\det(A_k)} - \det(aI_{n-k})\right)^{\frac{1}{n-k}} + \left(\frac{\det(B)}{\det(B_k)} - \det(bI_{n-k})\right)^{\frac{1}{n-k}},$$

with equality if and only if $a^{-1}A = b^{-1}B$.

The original proof of Theorem 1.1 seems to be lengthy. In this note, we first give a simple proof of Theorem 1.1 using Bellman's inequality. And then, we show some generalizations of the matrix form of the Brunn–Minkowski inequality to the case of matrices whose numerical ranges are contained in a sector.

2. Auxiliary results

Let $\mathbb{M}_n$ denote the set of $n \times n$ complex matrices. For two Hermitian matrices X and Y, we write $X \geq Y$ to mean that $X - Y$ is positive semidefinite, so $X \geq 0$ means that X is positive semidefinite. If X is positive definite, then we write $X > 0$. Let I_n denote the $n \times n$ identity matrix.

If $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathbb{M}_n$ with X_{11} nonsingular, then the Schur complement of X_{11} in X is defined as

$$X/X_{11} = X_{22} - X_{21}X_{11}^{-1}X_{12}.$$

For more information on the Schur complement, we refer to the comprehensive survey (see [17]).