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The Heisenberg group associated to a Hilbert space

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Let $(X, \langle\cdot\rangle)$ be a complex Hilbert space. The set $H_X = X \times \mathbb{R}$ equipped with the binary operation $(x_1, t_1) \cdot (x_2, t_2) = (x_1 + x_2, t_1 + t_2 + 2\operatorname{Im}(\langle x_1, x_2\rangle))$ is the famous Heisenberg group. For all $\alpha > 0$, $k > 0$ let $N_{\alpha,k} : H_X \to [0, \infty)$ be defined by $N_{\alpha,k}(x, t) = \left(\|x\|^{\frac{\alpha}{k}} + |t|^{\frac{\alpha}{2k}}\right)^{\frac{1}{\alpha}}$. We prove that

$$N_{\alpha,k}((x_1, t_1) \cdot (x_2, t_2)) \leq \left([N_{\alpha,k}(x_1, t_1)]^k + [N_{\alpha,k}(x_2, t_2)]^k\right)^{\frac{1}{k}}$$

if and only if $\alpha \geq 4k$. A similar result is proved for a real Hilbert space. Related questions are investigated.

1. Introduction and background

Let n be a natural number. Following E.M. Stein, see [10], the nth complex Heisenberg group H^{2n+1} is $\mathbb{C}^n \times \mathbb{R}$ equipped with the binary operation

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2\operatorname{Im}(\langle z, z'\rangle))$$

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where $\langle , \rangle$ is the complex scalar product on $\mathbb{C}^n$ and $\operatorname{Im}(a)$ denotes the imaginary part of a complex number a, see also [9]. On H^{2n+1} there is the Koranyi norm defined by $N(z,t) = \left(\|z\|^4 + t^2\right)^{\frac{1}{4}}$, see [2,4]. The nth real Heisenberg group H^{2n+1} is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ equipped with the binary operation

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(\langle y_1, x_2\rangle - \langle x_1, y_2\rangle))$$

where $\langle , \rangle$ is the scalar product on $\mathbb{R}^n$ and in this case the Koranyi norm is defined by $N(x,y,t) = \left(\left(\|x\|^2 + \|y\|^2\right)^2 + t^2\right)^{\frac{1}{4}}$, see e.g. [5]. In this paper we introduce the concept of the Heisenberg group associated to a complex or a real Hilbert space, define on them a kind of Koranyi norm, see Definitions 1, 2 and give conditions for this norm to be a group norm, see Theorems 1, 2. Let us recall the concepts, notion and notation used in this paper. Let $(G, \cdot)$ be a group with e the unit element and x^{-1} the inverse of an element $x \in G$. A function $N : G \to \mathbb{R}$ is called a group norm if: 1) $N(x) = 0$ if and only if $x = e$; 2) $N\left(x^{-1}\right) = N(x)$, $\forall x \in G$; 3) $N(xy) \le N(x) + N(y)$, $\forall x, y \in G$. In this case $(G, \cdot, N)$ is called a normed group. It is well-known that if, $N : G \to \mathbb{R}$ is a group norm, then $d : G \times G \to \mathbb{R}$ defined by $d(x,y) = N\left(xy^{-1}\right)$ is a metric, called the metric associated to the group norm N. Also, it is easy to prove that if, $(G, \cdot)$ is a group, $N : G \to \mathbb{R}$ is a function which satisfies the conditions 1), 2) and $d : G \times G \to \mathbb{R}$ is defined by $d(x,y) = N\left(xy^{-1}\right)$, then d satisfies the triangle inequality: $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in G$ if and only if $N(xy) \le N(x) + N(y)$, $\forall x, y \in G$. For $(a,b) \in \mathbb{R}^2$, $1 \le p \le \infty$ we consider the norm $\|(a,b)\|_p = (|a|^p + |b|^p)^{\frac{1}{p}}$ for $1 \le p < \infty$ and $\|(a,b)\|_\infty = \max(|a|, |b|)$. Note that $\lim_{p\to\infty} \|(a,b)\|_p = \|(a,b)\|_\infty$. If z is a complex number we denote by $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $|z|$ the real part, the imaginary part, the modulus of z. All notation and notion are standard, see [1].

2. The Heisenberg group associated to a complex Hilbert space

Following E.M. Stein, see [10, page 530], we introduce the Heisenberg group associated to a complex Hilbert space as follows.

Definition 1. Let $(X, \langle , \rangle)$ be a complex Hilbert space. The set $H_X = X \times \mathbb{R}$ equipped with the binary operation

$$(x_1, t_1) \cdot (x_2, t_2) = (x_1 + x_2, t_1 + t_2 + 2\operatorname{Im}(\langle x_1, x_2\rangle))$$

is a group, called the Heisenberg group associated to the complex Hilbert space X. For all $\alpha > 0$, $k > 0$ we define the (α, k)-Koranyi norm $N_{\alpha,k} : H_X \to [0, \infty)$ by the formula

$$N_{\alpha,k}(x,t) = \left(\|x\|^{\frac{\alpha}{k}} + |t|^{\frac{\alpha}{2k}}\right)^{\frac{1}{\alpha}}.$$