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Linear rank preservers of tensor products of rank one matrices

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ABSTRACT

Let $n_1, \ldots, n_k$ be integers larger than or equal to 2. We characterize linear maps $\phi : M_{n_1 \cdots n_k} \to M_{n_1 \cdots n_k}$ such that

$$\operatorname{rank}(\phi(A_1 \otimes \cdots \otimes A_k)) = 1$$

$$\text{whenever} \quad \operatorname{rank}(A_1 \otimes \cdots \otimes A_k) = 1$$

for all $A_i \in M_{n_i}$, $i = 1, \ldots, k$. Applying this result, we extend two recent results on linear maps that preserve the rank of special classes of matrices.

1. Introduction and statement of main results

Let $n \geq 2$ be positive integers. Denote by M_n the set of $n \times n$ complex matrices and $\mathbb{C}^n$ the set of complex column vectors with n components. Linear preserver problems concern

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the study of linear maps on matrices or operators with some special properties, which has a long history. In 1897, Frobenius [8] showed that a linear operator $\det(\phi(A)) = \det(A)$ for all $A \in M_n$ if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that ϕ has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN.$$

Since then, lots of linear preservers have been characterized, see [4,13] and their references. In particular, Marcus and Moyls [19] determined linear maps that send rank one matrices to rank one matrices, which have the form $A \mapsto MAN$ or $A \mapsto MA^TN$ for some nonsingular matrices M and N.

Recently, linear maps that preserve certain properties of tensor products are studied. The *tensor product* (*Kronecker product*) of two matrices $A \in M_m$ and $B \in M_n$ is defined to be $A \otimes B = [a_{ij}B]$, which is in M_{mn}. In [4], the authors determined linear maps on Hermitian matrices that leave the spectral radius of all tensor products invariant. In [3,5,6,14] the authors determined linear maps on M_{mn} that preserve Ky Fan norms, Shattern norms, numerical radius, k-numerical range, product numerical range of all matrices of the form $A \otimes B$ with $A \in M_m$ and $B \in M_n$. Notice that the set of matrices of tensor product form shares only a very small portion in M_{mn} and the sum of two tensor products is in general no longer a tensor product form. Therefore, such linear preserver problems are more challenging than the traditional problems. In some of the above mentioned papers, the authors have also extended their results to multipartite system, i.e., matrices of the form $A_1 \otimes \cdots \otimes A_k$ with $k \geq 2$.

In the literature, rank preserver problem is known to be one of the fundamental problems in this subject as many other preserver problems can be deduced to rank preserver problems. For example, the result of Marcus and Moyls [19] on linear rank one preservers has been applied in many other preserver results. More discussion can be found in [10]. Let $n_1, \ldots, n_k$ be positive integers of at least two. In [27], Zheng, Xu and Fošner showed that a linear map $\phi : M_{n_1 \cdots n_k} \to M_{n_1 \cdots n_k}$ satisfies

$$\operatorname{rank} \phi(A_1 \otimes \cdots \otimes A_k) = \operatorname{rank}(A_1 \otimes \cdots \otimes A_k) \quad \text{for all } A_i \in M_{n_i},\ i = 1, \ldots, k \tag{1.1}$$

if and only if ϕ has the form

$$\phi(A_1 \otimes \cdots \otimes A_k) = M(\psi_1(A_1) \otimes \cdots \otimes \psi_k(A_k))N \tag{1.2}$$

where $M, N \in M_{n_1 \cdots n_k}$ are nonsingular and ψ_i, $i = 1, \ldots, k$, is either the identity map or the transpose map. Their proof was done by induction on k with some smart argument on the rank of sum of certain matrices. The same authors also considered in [26] the injective maps on the space of Hermitian matrices satisfying (1.1) for rank one matrices only, the additive case was also considered in [16]. By using a structure theorem of Westwick [23], Lim [17] improved the result of Zheng et al. and showed that a linear map $\phi : M_{n_1 \cdots n_k} \to M_{n_1 \cdots n_k}$ satisfying (1.1) for rank one matrices and nonsingular matrices has the form (1.2) too.