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Instances of the Kaplansky–Lvov multilinear conjecture for polynomials of degree three

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ABSTRACT

Given a positive integer d, the Kaplansky–Lvov conjecture states that the set of values of a multilinear noncommutative polynomial $f \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ on the matrix algebra $M_d(\mathbb{C})$ is a vector subspace. In this article the technique of using one-wiggle families of Sylvester's clock-and-shift matrices is championed to establish the conjecture for polynomials f of degree three when d is even or $d < 17$.

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1. Introduction and the statement of the main result

Images of noncommutative polynomials play a fundamental role in noncommutative algebra and are a central topic of the theory of polynomial identities [18,19]. Another area where these objects play a prominent role is free analysis [23,11], especially its free real algebraic geometry branch [9]. This recent progress has led to a surge of interest in images of noncommutative polynomials in matrix rings. A fundamental open problem in this regard is (cf. [8]):

Conjecture 1.1 *(The Kaplansky–Lvov multilinear conjecture). Let f be a multilinear polynomial with complex coefficients, and let $d \in \mathbb{N}$. Then the set of values of f in $M_d(\mathbb{C})$ is a vector space.*

Conjecture 1.1 is stated in [8] for all fields, not just $\mathbb{C}$. We have, however, chosen to present the conjecture only over $\mathbb{C}$ as this is where our main interest lies. Likewise many of the results from the literature cited below were proved for large classes of fields, but we shall only state their restrictions to $\mathbb{C}$. Incidentally, by general model theory, all our results presented over $\mathbb{C}$ are valid over arbitrary algebraically closed fields of characteristic 0.

If the set of values of a noncommutative polynomial f is a vector subspace of $M_d(\mathbb{C})$, then it is a Lie ideal (see e.g. [4]), and hence either $\{0\}$, $\mathbb{C} \cdot I_d$, $M_d(\mathbb{C}) \cap \ker \operatorname{Tr} = [M_d(\mathbb{C}), M_d(\mathbb{C})]$ or $M_d(\mathbb{C})$ by an old result of Herstein. Thus noncommutative polynomials can be classified based on their (span of) values in $M_d(\mathbb{C})$. A very special instance of Conjecture 1.1 is the case $f = [x, y] = xy - yx$, which is a classical result in matrix theory due to Shoda [20] (see also Albert–Muckenhoupt [1]): every traceless matrix is a commutator. In [12] Kanel-Belov, Malev and Rowen established Conjecture 1.1 for $d = 2$, i.e., for values in 2×2 matrices. In [22] Špenko proves Conjecture 1.1 for *Lie* polynomials (i.e., elements of a free Lie algebra) of degree ≤ 4. Mesyan [17] extends this to polynomials of degree 3 which are sums of commutators, while Buzinski and Winstanley [5] present an extension to multilinear sums of commutators of degree 4. Further recent progress on images of multilinear polynomials is given in [2,7,15,16,13,14].

Our main result establishes Conjecture 1.1 for polynomials f of degree three when d is even or $d < 17$ is odd:

Theorem 1.2. *Let f be a complex multilinear polynomial of degree three.*

(1) *If $d \in \mathbb{N}$ is even then the image of f in $M_d(\mathbb{C})$ is a vector space.*
(2) *If $d \in \mathbb{N}$ is odd and $d < 17$, then the image of f in $M_d(\mathbb{C})$ is a vector space.*

The main novelty in our approach is the use of the clock-and-shift matrices first utilized by Sylvester [21]. These matrices are ubiquitous in mathematical physics (cf. [24, Chapter IV, §15] or [3,6]) and endow $M_d(\mathbb{C})$ with a group-with-cocycle structure (see