Linear Algebra and its Applications 502 (2016) 41–57

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Regularization matrices determined by matrix nearness problems

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A R T I C L E I N F O

Article history:
Received 31 January 2015
Accepted 8 December 2015
Available online 21 December 2015
Submitted by V. Mehrmann

MSC:
65R30
65F22
65F10

Keywords:
Tikhonov regularization
Regularization matrix
Matrix nearness problem

A B S T R A C T

This paper is concerned with the solution of large-scale linear discrete ill-posed problems with error-contaminated data. Tikhonov regularization is a popular approach to determine meaningful approximate solutions of such problems. The choice of regularization matrix in Tikhonov regularization may significantly affect the quality of the computed approximate solution. This matrix should be chosen to promote the recovery of known important features of the desired solution, such as smoothness and monotonicity. We describe a novel approach to determine regularization matrices with desired properties by solving a matrix nearness problem. The constructed regularization matrix is the closest matrix in the Frobenius norm with a prescribed null space to a given matrix. Numerical examples illustrate the performance of the regularization matrices so obtained.

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[1] Research supported by Fund of China Scholarship Council, the young scientific research backbone teachers of CDUT (KYGG201309).
[2] Research supported by a grant from Sapienza Università di Roma.
[3] Research supported in part by NSF grant DMS-1115385.

http://dx.doi.org/10.1016/j.laa.2015.12.008
0024-3795/

1. Introduction

We are concerned with the computation of an approximate solution of linear least-squares problems of the form

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \|K\boldsymbol{x}-\boldsymbol{b}\|, \qquad K\in\mathbb{R}^{m\times n}, \quad \boldsymbol{b}\in\mathbb{R}^m, \tag{1.1}$$

with a large matrix K with many singular values of different orders of magnitude close to the origin. In particular, K is severely ill-conditioned and may be singular. Linear least-squares problems with a matrix of this kind often are referred to as linear discrete ill-posed problems. They arise, for instance, from the discretization of linear ill-posed problems, such as Fredholm integral equations of the first kind with a smooth kernel. The vector $\boldsymbol{b}$ of linear discrete ill-posed problems that arise in applications typically represents measured data that is contaminated by an unknown error $\boldsymbol{e}\in\mathbb{R}^m$.

Let $\widehat{\boldsymbol{b}}\in\mathbb{R}^m$ denote the unknown error-free vector associated with $\boldsymbol{b}$, i.e.,

$$\boldsymbol{b}=\widehat{\boldsymbol{b}}+\boldsymbol{e}, \tag{1.2}$$

and let $\widehat{\boldsymbol{x}}$ be the solution of the unavailable linear system of equations

$$K\boldsymbol{x}=\widehat{\boldsymbol{b}}, \tag{1.3}$$

which we assume to be consistent. If K is singular, then $\widehat{\boldsymbol{x}}$ denotes the solution of minimal Euclidean norm.

Let $K^\dagger$ denote the Moore–Penrose pseudoinverse of K. The solution of minimal Euclidean norm of (1.1), given by

$$K^\dagger\boldsymbol{b}=K^\dagger\widehat{\boldsymbol{b}}+K^\dagger\boldsymbol{e}=\widehat{\boldsymbol{x}}+K^\dagger\boldsymbol{e},$$

typically is not a useful approximation of $\widehat{\boldsymbol{x}}$ due to severe propagation of the error $\boldsymbol{e}$. This depends on the large norm of $K^\dagger$. Therefore, one generally replaces the least-squares problem (1.1) by a nearby problem, whose solution is less sensitive to the error $\boldsymbol{e}$. This replacement is known as regularization. One of the most popular regularization methods is due to Tikhonov. This method replaces (1.1) by a penalized least-squares problem of the form

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{\|K\boldsymbol{x}-\boldsymbol{b}\|^2+\mu\|L\boldsymbol{x}\|^2\right\}, \tag{1.4}$$

where $L\in\mathbb{R}^{p\times n}$ is referred to as a regularization matrix and the scalar $\mu>0$ as a regularization parameter; see, e.g., [1,9,11]. Throughout this paper $\|\cdot\|$ denotes the Euclidean vector norm or the spectral matrix norm. We assume that the matrices K and L satisfy

$$\mathcal{N}(K)\cap\mathcal{N}(L)=\{\mathbf{0}\}, \tag{1.5}$$